The isoperimetric problem in general relativity

Paul Laurain- Institut Mathématiques de Jussieu (Paris 7)

Institut Henri Poincaré - Paris

February 16, 2018

contents

Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface.

Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface.

Definition

The quadratic form on $T_p\Sigma$ defined by

$$II_p(\vec{v}) := -\langle d\vec{N}_p(\vec{v}), \vec{v}\rangle,$$

is called the second fundamental quadratic form of Σ.

Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface.

Definition

The quadratic form on $T_p\Sigma$ defined by

$$II_p(\vec{v}) := -\langle d\vec{N}_p(\vec{v}), \vec{v}\rangle,$$

is called the second fundamental quadratic form of Σ.

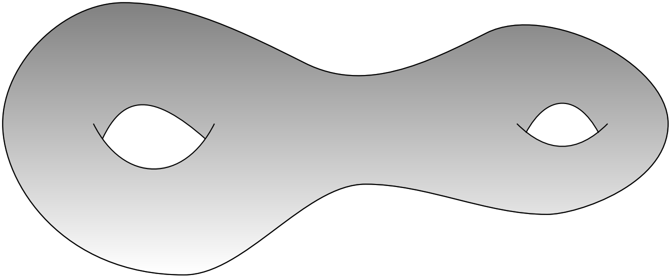
Definition

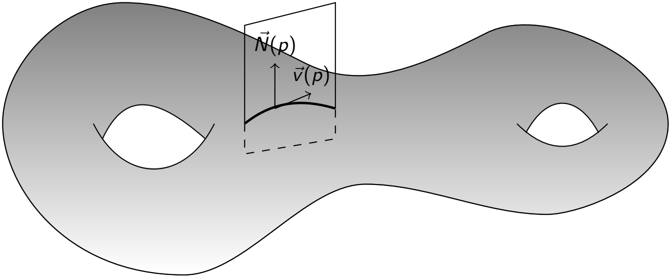
We define two maps K et H, namely the Gauss and the mean curvature, as follows

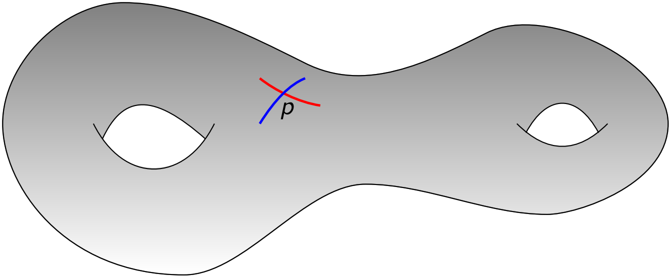
$$K(p) = det(d\vec{N}_p)$$

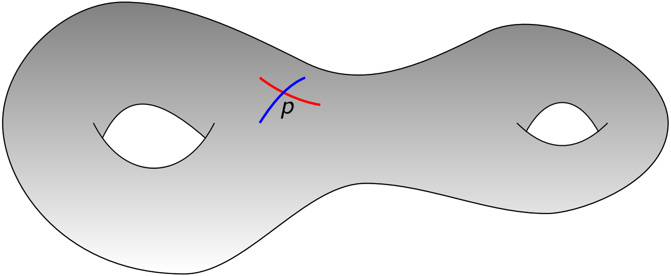
and

$$H(p) = \frac{1}{2}trace(d\vec{N}_p).$$

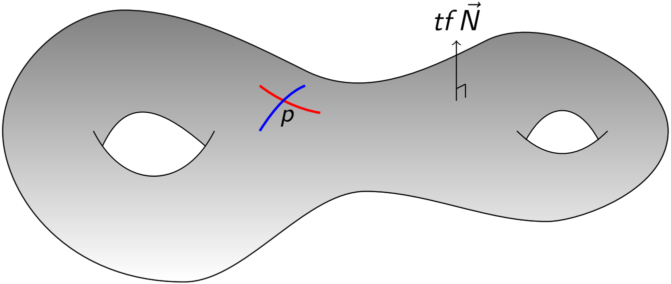



$\vec{N}(p)$
$\vec{v}(p)$

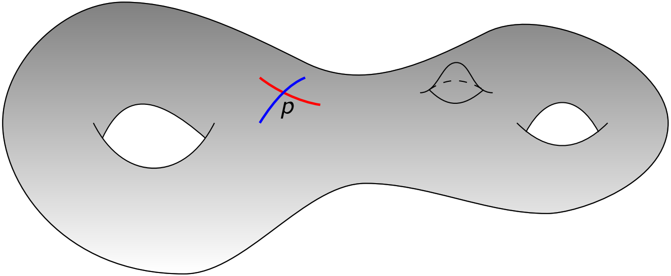

p



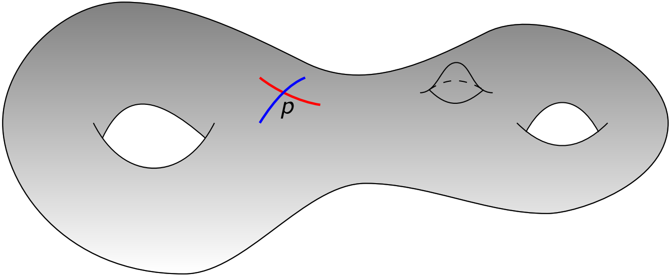
- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$



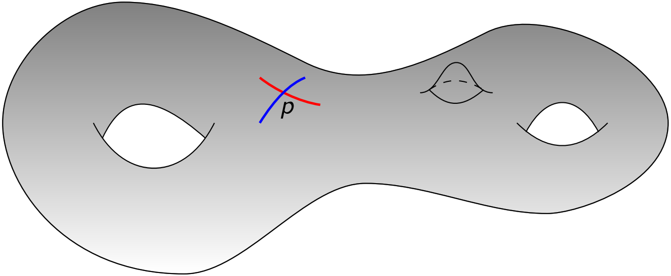
- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$



- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$



- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$
- $A'(0) = -2 \int_\Sigma f\, H\, d\Sigma$ et $V'(0) = \int_\Sigma f\, d\Sigma$



- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$
- $A'(0) = -2 \int_\Sigma f\, H\, d\Sigma$ et $V'(0) = \int_\Sigma f\, d\Sigma$

Theorem

Surfaces which minimize their area with a fixed volume(isoperimetric surfaces) have constant mean curvature.

Theorem (Hopf 1951)

Let S be a compact simply connected surface with constant mean curvature, then it is a round sphere.

Theorem (Hopf 1951)

Let S be a compact simply connected surface with constant mean curvature, then it is a round sphere.

Theorem (Aleksandrov 1956)

Let S be a compact embedded surface with constant mean curvature, then it is a round sphere

Theorem (Hopf 1951)

Let S be a compact simply connected surface with constant mean curvature, then it is a round sphere.

Theorem (Aleksandrov 1956)

Let S be a compact embedded surface with constant mean curvature, then it is a round sphere

This hypothesis are optimal.

Theorem (Wente 1983)

There exists an immersion of T^2 into $\mathbb{R}^3$ whose image has constant mean curvature.

Theorem (Hopf 1951)

Let S be a compact simply connected surface with constant mean curvature, then it is a round sphere.

Theorem (Aleksandrov 1956)

Let S be a compact embedded surface with constant mean curvature, then it is a round sphere

This hypothesis are optimal.

Theorem (Wente 1983)

There exists an immersion of T^2 into $\mathbb{R}^3$ whose image has constant mean curvature.

Kapouleas proved in the 90' that there are CMC surfaces of arbitrary genus.

Second variation of the area:

$$A''(0) = -\int_{\Sigma} f(\Delta f + \|II\|^2 f)d\Sigma.$$

Second variation of the area:

$$A''(0) = -\int_\Sigma f(\Delta f + \|II\|^2 f) d\Sigma.$$

A CMC is (weakly) **stable** if for all $f \in C_c^\infty(\Sigma)$ (with $\int f = 0$) then

$$\int_\Sigma \|II\|^2 f^2 d\Sigma \leq \int_\Sigma \|\nabla f\|^2 d\Sigma$$

Second variation of the area:

$$A''(0) = -\int_\Sigma f(\Delta f + \|II\|^2 f)d\Sigma.$$

A CMC is (weakly) **stable** if for all $f \in C_c^\infty(\Sigma)$ (with $\int f = 0$) then

$$\int_\Sigma \|II\|^2 f^2 d\Sigma \leq \int_\Sigma \|\nabla f\|^2 d\Sigma$$

Theorem (Fischer-Colobrie, Schoen, 82,Barbosa, Do Carmo, 84)

The only (weakly) stable CMC (complete) surfaces of $\mathbb{R}^3$ are planes and round spheres.

contents

Theorem (Ye, 1991)

Let $(\mathcal{N}, g)$ be a Riemannian manifold and $p \in \mathcal{N}$ a non-degenerate critical point of the scalar curvature. Then there exists a surface with constant mean curvature in every neighborhood of p.

Theorem (Ye, 1991)

Let $(\mathcal{N}, g)$ be a Riemannian manifold and $p \in \mathcal{N}$ a non-degenerate critical point of the scalar curvature. Then there exists a surface with constant mean curvature in every neighborhood of p.

We can relax the hypothesis of non-degeneracy for existence: Pacard et Xu 2009.

Theorem (Ye, 1991)

Let $(\mathcal{N}, g)$ be a Riemannian manifold and $p \in \mathcal{N}$ a non-degenerate critical point of the scalar curvature. Then there exists a surface with constant mean curvature in every neighborhood of p.

We can relax the hypothesis of non-degeneracy for existence: Pacard et Xu 2009.
This condition is also necessary, Laurain 2011.

Theorem (Johnson & Morgan, 2000)

Let $(\mathcal{N}, g)$ be a compact Riemannian manifold. Then isoperimetric domains of small volume are nearly round spheres.

Theorem (Johnson & Morgan, 2000)

Let $(\mathcal{N}, g)$ be a compact Riemannian manifold. Then isoperimetric domains of small volume are nearly round spheres.

Theorem (Druet, 2002)

Let $(\mathcal{N}, g)$ be a compact Riemannian manifold and Ω_V a sequence of isoperimetric domains of volume V, then

$$\Omega_V \to p \text{ as } V \to 0,$$

where p is a point of maximum of the scalar curvature.

The second variation of area is:

$$A''(0) = -\int_{\Sigma} f(\Delta f + (\|II\|^2 + Ricc(\vec{N}, \vec{N}))f)d\Sigma.$$

The second variation of area is:

$$A''(0) = -\int_{\Sigma} f(\Delta f + (\|II\|^2 + Ricc(\vec{N}, \vec{N}))f)d\Sigma.$$

Using Gauss equation it becomes

$$A''(0) = -\int_{\Sigma} f\left(\Delta f + \left(\frac{1}{2}(\|II\|^2 + R - \frac{K}{2}\right) f\right) d\Sigma,$$

where R is the scalar curvature and K the Gaussian curvature.

The second variation of area is:

$$A''(0) = -\int_\Sigma f(\Delta f + (\|II\|^2 + Ricc(\vec{N}, \vec{N}))f)d\Sigma.$$

Using Gauss equation it becomes

$$A''(0) = -\int_\Sigma f\left(\Delta f + \left(\frac{1}{2}(\|II\|^2 + R - \frac{K}{2}\right) f\right) d\Sigma,$$

where R is the scalar curvature and K the Gaussian curvature. Hence when **the scalar curvature is non-negative** we can derive some topological constraint on Σ. Ror instance,in the compact case, if $H = 0$ (stable) the genus is smaller than 1 (Schoen-Yau), if H is large enough (weakly stable) the genus is smaller than 3 (Rosenberg).

contents

General Relativity postulates:

- The space-time is a $(3,1)$ Lorentzian Manifold $(\tilde{M}, \tilde{g})$.
- Free particles travel along time-geodesic.
- $\tilde{g}$ satisfies the Einstein equation:

$$\tilde{Ric} - \frac{\tilde{R}}{2}\tilde{g} = 8\pi T,$$

where T is energy momentum tensor.

General Relativity postulates:

- The space-time is a $(3,1)$ Lorentzian Manifold $(\tilde{M}, \tilde{g})$.
- Free particles travel along time-geodesic.
- $\tilde{g}$ satisfies the Einstein equation:

$$\tilde{Ric} - \frac{\tilde{R}}{2}\tilde{g} = 8\pi T,$$

  where T is energy momentum tensor.

What are the possible Universe (space slice (M, g))?
To simplify we consider some time-symmetric space-time which is isolated.

General Relativity postulates:

- The space-time is a $(3,1)$ Lorentzian Manifold $(\tilde{M}, \tilde{g})$.
- Free particles travel along time-geodesic.
- $\tilde{g}$ satisfies the Einstein equation:

$$\tilde{Ric} - \frac{\tilde{R}}{2}\tilde{g} = 8\pi T,$$

where T is energy momentum tensor.

What are the possible Universe (space slice (M, g))?
To simplify we consider some time-symmetric space-time which is isolated.
So Mathematically the question is:
What are the asymptotically flat 3-Riemannian manifolds with non-negative scalar curvature?

Definition

Let (M, g) be a 3-manifold, it is said to be Asymptotically Flat (AF) (with one end), if there exists a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and in those coordinates

$$g = \delta^{ij} + O_2(|x|^{-\tau}),$$

with $\tau > \frac{1}{2}$.

Definition

Let (M, g) be a 3-manifold, it is said to be Asymptotically Flat (AF) (with one end), if there exists a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and in those coordinates

$$g = \delta^{ij} + O_2(|x|^{-\tau}),$$

with $\tau > \frac{1}{2}$.

Theorem (Arnowitt, Deser, Misner, 61 , Bartnik, Chrusciel, 80')

Let (M, g) be an asymptotically flat manifold such that $R \in L^1$, then the following limit exists

$$\lim_{R \to +\infty} \frac{1}{16\pi} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j \, d\sigma,$$

moreover it depends only on the metric. Let denote it m for mass.

Theorem (Schoen-Yau, 79)

Let (M, g) be an AF manifold with nonnegative scalar curvature. Then $m \geq 0$ with equality if and only if M is isometric to $\mathbb{R}^3$.

The unique rotationally invariant solution of Einstein-Equation is given by the space slice $\mathbb{R}^3 \setminus \{0\}$, $\left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ is an AF manifold with vanishing scalar curvature and mass m. It is the Schwarzschild metric.

The unique rotationally invariant solution of Einstein-Equation is given by the space slice $\mathbb{R}^3 \setminus \{0\}$, $\left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ is an AF manifold with vanishing scalar curvature and mass m. It is the Schwarzschild metric.

Definition

A 3-manifold (M, g) is said to be Schwarzschildean (with one end), if there exists a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and in those coordinates

$$g = \left(1 + \frac{m}{2|x|}\right)^4 \delta^{ij} + O_2(|x|^{-2}).$$

The unique rotationally invariant solution of Einstein-Equation is given by the space slice $\mathbb{R}^3 \setminus \{0\}$, $\left(1+\frac{m}{2|x|}\right)^4 \delta_{ij}$ is an AF manifold with vanishing scalar curvature and mass m. It is the Schwarzschild metric.

Definition

A 3-manifold (M, g) is said to be Schwarzschildean (with one end), if there exists a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0,1)$ and in those coordinates

$$g = \left(1+\frac{m}{2|x|}\right)^4 \delta^{ij} + O_2(|x|^{-2}).$$

The mass is unchanged by translation, considering $\mathbb{R}^3 \setminus \{p\}$, $\left(1+\frac{m}{2|x-p|}\right)^4 \delta_{ij}$.
But can we detect the "center" of this translated version of Schwarzschild?

Definition

Let (M, g) be an AF manifold, such that

$$|g_{ij} - \delta_{ij}| + |x||\Gamma_{ij}^k| + |x|^2|Ric_{ij}| + |x|^{\frac{5}{2}}|S| \leq \frac{C}{|x|^{\frac{1}{2}+\varepsilon}}.$$

Then it satisfies the **weak Regge-Teitelboim** *condition, if*

$$|g(x) - g(-x)| + |x||\Gamma(x) + \Gamma(-x)| \leq \frac{C}{|x|^{1+\varepsilon}}.$$

It satisfies the **strong Regge-Teitelboim** *condition, if*

$$|g(x) - g(-x)| + |x||\Gamma(x) + \Gamma(-x)| + |x|^2|Ric(x) - Ric(-x)| + |x|^{\frac{5}{2}}|S(x) - S(-x)| \leq \frac{C}{|x|^{\frac{3}{2}+\varepsilon}}.$$

Theorem (Beig, O'Murchadha, 90)

Let (M, g) an AF manifold satisfying the strong RT condition, **with non vanishing mass**, *then the following limit exists*

$$\lim_{R\to+\infty} \frac{1}{16\pi m} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j x^\alpha - (g_{i\alpha}\nu^i - g_{ii}\nu^\alpha)\, d\sigma,$$

moreover it depends only on the metric. Let denote it C^α_{ADM} for the center of mass.

Theorem (Beig, O'Murchadha, 90)

Let (M, g) an AF manifold satisfying the strong RT condition, **with non vanishing mass**, *then the following limit exists*

$$\lim_{R \to +\infty} \frac{1}{16\pi m} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j x^\alpha - (g_{i\alpha}\nu^i - g_{ii}\nu^\alpha)\, d\sigma,$$

moreover it depends only on the metric. Let denote it C^α_{ADM} for the center of mass.

The strong RT condition has been proved to be optimal by Cederbaum & Nerz (13): Constructing metric with divergent center of mass.

contents

Theorem (Christodoulou-Yau, 88)

Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma H^2 \, d\sigma\right)$$

of a closed stable CMC is non negative.

Theorem (Christodoulou-Yau, 88)

Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma H^2 \, d\sigma\right)$$

of a closed stable CMC is non negative.

Theorem (Huisken-Yau 96, Ye 97)

Let M be a Schwarzschildean manifold with positive mass, then for R large enough we can perturb the sphere $S(0, R)$ into a stable CMC surface Σ_R. Those spheres form a foliation.

Theorem (Christodoulou-Yau, 88)

Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma H^2 \, d\sigma \right)$$

of a closed stable CMC is non negative.

Theorem (Huisken-Yau 96, Ye 97)

Let M be a Schwarzschildean manifold with positive mass, then for R large enough we can perturb the sphere $S(0, R)$ into a stable CMC surface Σ_R. Those spheres form a foliation.

Improvements: L.H. Huang, J. Metzger and finally C. Nerz(14) who prove the existence into an AF manifold.

Using this foliation Σ_R, you can take the following limit

$$C_{HY}^{\alpha} = \lim_{R\to\infty} \frac{\int_{\Sigma_R} x^{\alpha}\, d\sigma}{\int_{\Sigma_R} d\sigma},$$

As **a new definition of center of mass.**

Using this foliation Σ_R, you can take the following limit

$$C^{\alpha}_{HY} = \lim_{R\to\infty} \frac{\int_{\Sigma_R} x^{\alpha}\, d\sigma}{\int_{\Sigma_R} d\sigma},$$

As **a new definition of center of mass.**

In *Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1911–1946.* , Nerz proved, **under the weak RT** conditions that, as soon one definition is valid, the second one is also and

$$C_{HY} = C_{ADM}.$$

Using this foliation Σ_R, you can take the following limit

$$C^{\alpha}_{HY} = \lim_{R\to\infty} \frac{\int_{\Sigma_R} x^{\alpha}\, d\sigma}{\int_{\Sigma_R} d\sigma},$$

As **a new definition of center of mass.**

In *Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1911–1946.* , Nerz proved, **under the weak RT** conditions that, as soon one definition is valid, the second one is also and

$$C_{HY} = C_{ADM}.$$

The foliation provide also some kind of intrinsic coordinates, what about uniqueness?

Theorem (Qing, Tian, 07)

Let (M, g) a Schwarzschildean manifold with positive mass. Then exists a compact set K such that stable CMC spheres which separates the infinity from K coincide with the leafs of the CMC foliation.

Theorem (Qing, Tian, 07)

Let (M, g) a Schwarzschildean manifold with positive mass. Then exists a compact set K such that stable CMC spheres which separates the infinity from K coincide with the leafs of the CMC foliation.

Improvements:
Ma (10) AF+ Strong RT, Ma(16) $|g - \delta| = O_4(r^{-1})$ and $|S| = O(r^{-3+\varepsilon})$, Laurain-Metzger(17 **under the weak RT**.

A sequence of spheres which does not separate a compact set from infity either **intersect a compact region** or is **outlying**.

Theorem (Carlotto, Chodosh and Eichmair,16)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature. Then for every compact K there exists $\alpha_K > 0$ such that if Σ is a stable constant mean curvature surfaces then, with $|\Sigma| \geq \alpha_K$, then Σ is disjoint from K.

Theorem (Carlotto, Chodosh and Eichmair,16)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature. Then for every compact K there exists $\alpha_K > 0$ such that if Σ is a stable constant mean curvature surfaces then, with $|\Sigma| \geq \alpha_K$, then Σ is disjoint from K.

Consequence: to get global uniqueness, we need to exclude the outlying case .

Theorem (Carlotto, Chodosh and Eichmair,16)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature. Then for every compact K there exists $\alpha_K > 0$ such that if Σ is a stable constant mean curvature surfaces then, with $|\Sigma| \geq \alpha_K$, then Σ is disjoint from K.

Consequence: to get global uniqueness, we need to exclude the outlying case .
It is a consequence of the following quantitative version of the Positive Mass Theorem

Theorem (Carlotto, Chodosh and Eichmair,16)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature and which has horizon boundary. The the only complete stable minimal embeddings are embeddings of components of the horizon.

The assumption on the positive mass and the Schwarschildean asymptotic flatness can't be drop due to the following theorem

The assumption on the positive mass and the Schwarschildean asymptotic flatness can't be drop due to the following theorem

Theorem (Carlotto, Schoen, 16)

Given a scalar-flat asymptotically flat metric g there exist cones and scalar-flat asymptotically flat metrics which coincides with g inside of the cones and are **flat** *outside slightly larger cones.*

The assumption on the positive mass and the Schwarschildean asymptotic flatness can't be drop due to the following theorem

Theorem (Carlotto, Schoen, 16)

Given a scalar-flat asymptotically flat metric g there exist cones and scalar-flat asymptotically flat metrics which coincides with g inside of the cones and are **flat** *outside slightly larger cones.*

Here the decay of metric is $\tau \in (1/2, 1)$.

The assumption on the positive mass and the Schwarschildean asymptotic flatness can't be drop due to the following theorem

Theorem (Carlotto, Schoen, 16)

Given a scalar-flat asymptotically flat metric g there exist cones and scalar-flat asymptotically flat metrics which coincides with g inside of the cones and are **flat** *outside slightly larger cones.*

Here the decay of metric is $\tau \in (1/2, 1)$.
See, *Anti-gravity à la Carlotto-Schoen*, Piotr T. Chruściel, Séminaire Bourbaki.

Theorem (Brendle, Eichmair, 14)

There is a complete Riemannian 3-manifold (M, g) *that is Schwarschildean with mass* $m > 0$ *which admits a sequence of arbitrary large outlying stable constant mean curvature surfaces* Σ_k.

Theorem (Brendle, Eichmair, 14)

There is a complete Riemannian 3-manifold (M, g) *that is Schwarschildean with mass* $m > 0$ *which admits a sequence of arbitrary large outlying stable constant mean curvature surfaces* Σ_k.

Theorem (Brendle and Eichmair, 14)

Let (M, g) *be a complete Riemannian 3-manifold that is Schwarschildean with mass* $m > 0$ *in the following sense*

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij} + T_{ij} + o_4(r^{-2})$$

where T *is an homogeneous tensor of degree* -2. *If the scalar curvature satisfy* $R \geq -o(r^{-4})$, *there is no sequence of outlying stable constant mean curvature surfaces* Σ_k *such that* $\lim_k d(\Sigma_k, 0) H_k \in (0, \infty)$.

Theorem (Chodosh and Eichmair,17)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature. We fix K a compact set, then there exists $\eta > 0$, such that for every outlying stable constant mean curvature surface Σ, we have

$$d(\Sigma, K)H_\Sigma \geq \eta.$$

Theorem (Chodosh and Eichmair,17)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature. We fix K a compact set, then there exists $\eta > 0$, such that for every outlying stable constant mean curvature surface Σ, we have

$$d(\Sigma, K)H_\Sigma \geq \eta.$$

To get global uniqueness it suffices to exclude outlying surface whose distance to origin is much bigger than the diameter.

Theorem (Chodosh and Eichmair,17)

Let (M, g) be a complete 3 Manifold, Schwarschildean with positive mass and with none negative curvature, which satisfies either

$$R \equiv 0$$

or

$$x^i x^j \partial_i \partial_j R \geq 0 \text{ outside a compact set}$$

Then any stable constant mean curvature surface with area large enough is part of the canonical foliation.

Theorem (O. Chodosh, M. Eichmair, Y. Shi, and H. Yu , 16)

Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat and which has non-negative scalar curvature. Unless (M, g) is isometric to flat $\mathbb{R}^3$, for every sufficiently large $V > 0$, there is a unique surface of least area that encloses volume V in (M, g). This surface is a leaf of the canonical foliation.

Theorem (O. Chodosh, M. Eichmair, Y. Shi, and H. Yu , 16)

Let (M, g) be a complete Riemannian 3-manifold that is asymptotically flat and which has non-negative scalar curvature. Unless (M, g) is isometric to flat $\mathbb{R}^3$, for every sufficiently large $V > 0$, there is a unique surface of least area that encloses volume V in (M, g). This surface is a leaf of the canonical foliation.

The tools to prove this theorem is the study of the behavior of the Hawking mass, which is in a certain sense encode the defect of the isoperimetric ratio, this idea was introduce by Bray.

Theorem (Bray, 98)

In the exact Schwarzschild geometry, with $m > 0$, the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume(separating them from the horizon).

Theorem (Bray, 98)

In the exact Schwarzschild geometry, with $m > 0$, the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume(separating them from the horizon).

Theorem (Brendle, 13)

Every closed embedded constant mean curvature surface in the exact Schwarzschild, with $m > 0$, geometry is a spherically symmetric sphere.

Theorem (Bray, 98)

In the exact Schwarzschild geometry, with $m > 0$, the spherically symmetric spheres minimize the area among all other surfaces in their homology class containing the same volume(separating them from the horizon).

Theorem (Brendle, 13)

Every closed embedded constant mean curvature surface in the exact Schwarzschild, with $m > 0$, geometry is a spherically symmetric sphere.

Note that this result does not require the surfaces to be large or stable. It can be seen as a very general version of the Alexandrov theorem, since it also holds to be true in some general wrapped product with non-negative "curvature".

contents

Why about about maximizing the Hawking mass with respect to fixed area?

Why about about maximizing the Hawking mass with respect to fixed area?

- It seems more "physical".

Why about about maximizing the Hawking mass with respect to fixed area?

- It seems more "physical".
- Willmore surfaces.

Why about about maximizing the Hawking mass with respect to fixed area?

- It seems more "physical".
- Willmore surfaces.
- A new quasi-local mass ?

Thank you for your attention!