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SINGULARITIES AND WEAK CURVATURE BOUNDS

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Contents

Singularités et contraintes de courbure faibles

Le présent mémoire d'habilitation vise à décrire les travaux de recherche que j'ai effectués après mon doctorat, dans le domaine de l'analyse géométrique. Je m'intéresse à l'étude géométrique, analytique et topologique d'espaces singuliers pour lesquels on dispose d'une information sur la courbure, en me servant d'outils qui proviennent de l'analyse, de la géométrie riemannienne et de la géométrie métrique, comme les opérateurs de Schrödinger, le noyau de la chaleur, la convergence de Gromov-Hausdorff, les espaces de Dirichlet, l'inégalité de Bochner.

Une question naturelle au cœur de la géométrie moderne est celle de comprendre les conséquences d'une borne sur la courbure. D'importants résultats de géométrie riemannienne pour les variétés lisses, et plus récemment dans le cadre des espaces métriques ou métriques mesurés, ont permis d'obtenir une bonne compréhension des implications d'une minoration de la courbure sectionnelle ou de Ricci. Néanmoins, dans de nombreuses situations géométriques, un contrôle de l'une de ces deux courbures n'est pas satisfait et représenterait une hypothèse supplémentaire trop restrictive. C'est le cas par exemple dans l'étude des flots géométriques et de leurs singularités, dans celle des problèmes variationnels de minimisation, ou encore dans l'étude des espaces de modules. De plus, dans ces problèmes, un passage à la limite dans la topologie appropriée est souvent nécessaire, ce qui engendre généralement des singularités. Un problème fondamental de la géométrie actuelle consiste donc à comprendre des bornes de courbure moins contraignantes et à élargir le cadre des objets étudiés, en incluant des variétés singulières. Une motivation supplémentaire pour l'étude des singularités, et en particulier des singularités coniques, est qu'en plus de représenter un sujet intéressant en lui-même, elles jouent un rôle central dans les résolutions récentes de plusieurs conjectures géométriques. Par exemple, la preuve de la géométrisation des variétés de dimension 3 conjecturée par W. Thurston nécessite l'étude de singularités orbifolds et des singularités du flot de Ricci, comme montré par G. Perelman. L'existence de métriques de Kähler-Einstein à singularités coniques le long d'une sous-variété de codimension 2 est cruciale pour la démonstration de l'existence de métriques de Kähler-Einstein sur une variété de Fano par X. X. Chen, S. Donaldson et S. Sun.

Mes travaux se situent dans ce contexte et visent à élargir la compréhension d'objets singuliers, provenant de limites de variétés ou non, en présence d'une contrainte sur la courbure scalaire ou sur celle de Ricci. Il s'agit de contraintes “faibles” en plusieurs sens. Premièrement, la courbure scalaire est la plus faible parmi les courbures riemanniennes : sous un contrôle de la courbure scalaire, de nombreuses questions restent ouvertes à la fois dans le cadre lisse et singulier. Pour ce qui concerne la courbure de Ricci, je m'intéresse d'un coté à une notion généralisée aux espaces métriques mesurés de courbure de Ricci minorée, et d'un autre à des conditions intégrales qui affaiblissent cette minoration pour étudier des limites de Gromov-Hausdorff de variétés lisses.

Je donne ici une présentation (presque) chronologique de mes travaux principaux : le point de départ de ma recherche a été l'étude d'un problème d'analyse géométrique, le problème de Yamabe, sur des variétés à singularités coniques itérées. Certains résultats que j'ai prouvés dans ce cadre, sous l'hypothèse que la courbure de Ricci est minorée en dehors des singularités, m'ont menée ensuite vers l'étude de la théorie des espaces RCD, c'est-à-dire des espaces métriques mesurés pour lesquels on dispose d'une notion synthétique de courbure de Ricci minorée et de dimension majorée, due à J. Lott, K. T. Sturm et C. Villani. Une minoration synthétique de la courbure de Ricci avait été préconisée par J. Cheeger suite à ses travaux en collaboration avec T. H. Colding sur les limites de Gromov-Hausdorff de variétés lisses à courbure de Ricci minorée : je me suis donc naturellement intéressée à ce qu'on appelle désormais la théorie de Cheeger-Colding. Cela m'a permis d'étudier des contraintes de courbure intégrales et plus faibles, les bornes de Kato, pour des suites de variétés lisses. Chaque chapitre de ce mémoire est dédié à l'un de ces axes de recherche : je présente brièvement comment mes travaux s'inscrivent dans la littérature, mes résultats principaux et les stratégies de leurs preuves, en illustrant des perspectives de recherche pour le travail futur. Je donne relativement peu de préliminaires, supposant le lecteur à l'aise avec les notions de base de géométrie riemannienne et d'analyse sur les variétés.

Dans le premier chapitre, je décris les résultats obtenus dans [Mon18] et en collaboration avec K. Akutagawa dans [AM22] concernant le problème de Yamabe sur les variétés à singularités coniques itérées, ou autrement dit, les espaces stratifiés. Ce problème consiste à trouver une métrique à courbure scalaire constante dans la classe conforme d'une métrique fixée. Une métrique conforme qui minimise la courbure scalaire totale, c'est-à-dire l'intégrale de la courbure scalaire sur la variété, parmi les métriques conformes de volume unitaire est une métrique à courbure scalaire constante et s'appelle métrique de Yamabe. Dans le cas des variétés compactes lisses, une métrique de Yamabe existe toujours, et toute métrique d'Einstein est une métrique de Yamabe. Pour les espaces stratifiés, la situation est plus complexe : dans [Mon18], j'ai montré comment certains résultats connus pour les variétés lisses restent vrais si l'angle le long des singularités de codimension 2 est inférieur à 2π. Si l'angle est supérieur à 2π, nous avons construit avec K. Akutagawa [AM22] des exemples de métriques d'Einstein singulières qui ne sont pas minimisantes et qui ne contiennent dans leur classe conforme aucune métrique de Yamabe.

Le deuxième chapitre de ce mémoire est dédié à certains aspects géométriques des espaces RCD. La théorie des bornes de Ricci synthétiques a débuté avec l'introduction de la condition de courbure-dimension $\mathrm{CD}(K, N)$ par K. T. Sturm et J. Lott et C. Villani à l'aide du transport optimal de mesures. L. Ambrosio, N. Gigli et G. Savaré ont ensuite peaufiné cette définition en introduisant les espaces RCD, qui satisfont une hypothèse analytique supplémentaire excluant les variétés Finsleriennes. De plus, dans les espaces RCD, une inégalité de Bakry-Émery, qui étend l'inégalité classique de Bochner est vérifiée. Dans ce cadre, je me suis intéressée en particulier à créer un pont entre la théorie RCD et l'étude des variétés singulières, en fournissant une classe d'exemples d'espaces stratifiés qui sont RCD, dans un travail en commun avec J. Bertrand, C. Ketterer et T. Richard [BKMR21]. La théorie des espaces RCD a été très récemment utilisée dans l'étude des variétés singulières qui ne sont pas forcément limites de variétés, par exemple dans les travaux récents de G. Székelyhidi [Sze24], ou X. Dai, Y. Sun et C. Wang [DSW24b] et X. Dai, C. Wang, L. Wang et G. Wei [DWWW24]. Dans ce chapitre, je présente également un résultat de stabilité du tore obtenu en collaboration avec A. Mondino et R. Perales [MMP22], qui éclaire certaines propriétés topologiques des espaces RCD.

La dernier chapitre présente mes travaux en collaboration avec G. Carron et D. Tewodrose concernant les limites de Gromov-Hausdorff de variétés dont la courbure de Ricci satisfait des bornes de Kato. Dans le cas ou la courbure de Ricci est minorée, les espaces limites obtenus par convergence de Gromov-Hausdorff ont été l'objet de l'étude de nombreux mathématiciens dès la démonstration du théorème de pré-compacité de M. Gromov dans les années 1980. Très récemment, J. Cheeger, W. Jiang et A. Naber ont résolu de nombreuses conjectures qui avaient été formulées dans ce domaine au cours des années 1990. Comme expliqué ci-dessus, l'hypothèse de courbure de Ricci minorée peut être trop contraignante dans certaines situations, et le but de notre travail avec G. Carron et D. Tewodrose est d'affaiblir cette condition tout en obtenant des résultats de structure et de régularité sur les limites. Une façon de considérer la minoration sur la courbure de Ricci consiste à observer qu'en coordonnées harmoniques, le tenseur de Ricci peut s'écrire comme l'opposé du Laplacien de la métrique, auquel s'ajoutent des termes d'erreurs quadratiques en la métrique et ses dérivées. Une minoration de la courbure de Ricci par une constante correspond donc, en termes analytiques, à une majoration du Laplacien de la métrique. Il est alors naturel de remplacer l'opérateur laplacien par un opérateur de Schrödinger, c'est-à-dire le laplacien auquel on retranche un potentiel choisi de façon à contenir les informations nécessaires sur la courbure. Dans [CMT24, CMT22, CMT23b, CMT23a] nous avons utilisé comme potentiel la partie négative de la courbure de Ricci et supposé qu'elle vérifie une condition inspirée par les potentiels de Kato dans $\mathbb{R}^n$, tels que définis par T. Kato et étudiés par B. Simon [Kat72, Sim82]. Nous avons ainsi obtenu une théorie de la régularité qui englobe la célèbre théorie de Cheeger-Colding pour les limites à courbure de Ricci minorée et dont je présente ici les grandes lignes.

Singularities and weak curvature bounds

The present memoir aims to describe the research I conducted after my Ph.D., in the domain of geometric analysis. I am interested in the geometric, analytic and topological study of singular spaces for which a control on the curvature is assumed, using tools that come from analysis, Riemannian geometry and metric geometry, such as Schrödinger operators, the heat kernel, Gromov-Hausdorff convergence, Dirichlet spaces, Bochner inequality.

A natural question at the heart of modern geometry consists in understanding the consequences of a bound on the curvature. Important results of Riemannian geometry for smooth manifolds, and more recently in the settings of metric and metric measure spaces, have provided a good comprehension of the implications of a lower bound on the sectional or Ricci curvature. Nevertheless, in many geometric situations in the current research, a bound on one of those curvatures is not satisfied and would represent a too restrictive additional assumption. This is the case for instance in the study of geometric flows and of their singularities, in the study of variational minimization problems, or in the one of moduli spaces. Moreover, in these problems, it is often necessary to use convergence in the appropriate topology, and this leads to singularities. Therefore, a fundamental question in recent geometry is to understand less restrictive bounds on the curvature and to enlarge the setting of the objects that are taken into account, by including singular manifolds. A further motivation for the study of singularities, and in particular of conical singularities, in addition to being an interesting subject of study on its own, is that they play a central role in the recent resolution of several geometric conjectures. For instance, the proof of Geometrization of 3-manifolds conjectured by W. Thurston uses orbifolds singularities and the study of singularities of the Ricci flow by G. Perelman. The starting point for the demonstration of the existence of Kähler-Einsten metrics on Fano manifolds due to X. X. Chen, S. Donaldson et S. Sun is the existence of Kähler-Einsten metrics with conical singularities along a codimension 2 submanifold.

My work belongs to this context and aims to enlarge the understanding of singular objects, either coming from limits of smooth manifolds or not, in presence of a bound on the scalar or Ricci curvature. These are "weak" curvature bounds in several meanings. First of all, scalar curvature is the weakest among Riemannian curvatures: under a bound on the scalar curvature, many questions are still open, both in the smooth and singular settings. As for the Ricci curvature, I am interested on one hand in a generalized notion of a lower Ricci bound for metric measure spaces, and on the other hand in integral conditions that weaken this lower bound in order to study Gromov-Hausdorff limits of smooth manifolds.

I give here a (mainly) chronological presentation of the principal elements of my work: the starting point of my research was the study of a geometric analysis problem, the Yamabe problem, in presence of iterated conical singularities. Some of the results that I have proven in this setting, under the assumption that the Ricci curvature is bounded from below away from the singularities, led me to study the theory of RCD spaces, that is, metric measure spaces for which a synthetic notion of a Ricci lower bound and an upper bound on the dimension is known, thanks to the work of J. Lott, K. T. Sturm and C. Villani. The definition of a generalized lower bound for the Ricci curvature was encouraged by J. Cheeger, motivated by his work in collaboration with T. H. Colding on Gromov-Hausdorff limits of smooth manifolds with Ricci curvature bounded from below: I got naturally interested in the so-called Cheeger-Colding theory. This led me to study weaker integral curvature bounds, Kato bounds, for sequences of smooth manifolds. Each chapter of this memoir is devoted to one of these research directions: I briefly present how my work relates to the literature, my main results and the strategies of their proofs, and illustrate some perspectives for future developments. I give relatively few preliminaries, assuming the reader to be familiar with basics notions of Riemannian geometry and analysis on manifolds.

In the first chapter, I describe the results obtained in [Mon18] and in collaboration with K. Akutagawa [AM22] concerning the Yamabe problem on manifolds with iterated conical singularities, that is, stratified spaces. This problem consists in finding a particular metric of constant scalar curvature in the conformal class of a given metric. A conformal metric which minimizes the total scalar curvature among the unit volume conformal metrics has constant scalar curvature and is called a Yamabe metric. In the case of compact smooth manifolds, a Yamabe metric always exists, and an Einstein metric is a Yamabe metric. As for stratified spaces, the situation is more involved: in [Mon18], I showed how some results for smooth manifolds hold in presence of singularities if the angle along the codimension 2 stratum is not larger than 2π. If the angle is larger than 2π, with K. Akutagawa in [AM22] we constructed examples of singular Einstein metrics that are not minimizing and do not contain any Yamabe metric in their conformal class.

The second chapter of this memoir is devoted to geometric aspects of RCD spaces. The theory of synthetic Ricci lower bounds started with the introduction of the curvature-dimension condition $\mathrm{CD}(K,N)$ by K. T. Sturm and J. Lott and C. Villani, using optimal transport of measures. L. Ambrosio, N. Gigli and G. Savaré later refined this definition by introducing RCD spaces, which satisfy an additional analytic assumption excluding Finsler manifolds. Moreover, RCD spaces carry a Bakry-Émery inequality, which extends the classical Bochner inequality. In this setting, I got interested in particular in creating a bridge between the RCD theory and the study of singular manifolds, by giving a class of examples of stratified spaces that are RCD, in a collaboration with

J. Betrand, C. Ketterer and T. Richard [BKMR21]. The theory of RCD spaces has been very recently applied to singular manifolds that are not necessarily limits of smooth manifolds, for instance in the recent work by G. Székelyhidi [Sze24], or by X. Dai, Y. Sun et C. Wang [DSW24b] and by X. Dai, C. Wang, L. Wang et G. Wei [DWWW24]. I also present in this chapter a torus stability result obtained in collaboration with A. Mondino and R. Perales [MMP22], which enlightens some topological properties of RCD spaces.

The last chapter presents my work in collaboration with G. Carron and D. Tewodrose concerning Gromov-Hausdorff limits of manifolds whose Ricci curvature satisfies a Kato bound. In the case of a lower bound on Ricci, the limit spaces obtained by Gromov-Hausdorff convergence have been the object of study of many mathematicians since the proof of M. Gromov's pre-compactness theorem in the 1980s. Very recently, J. Cheeger, W. Jiang and A. Naber solved many of the conjectures that had been formulated in this domain during the 1990s. As explained above, the assumption of having a uniform lower Ricci bound can be too restrictive in several geometric situations, and the goal of our work with G. Carron and D. Tewodrose is to weaken this hypothesis while still obtaining structure and regularity results for the limit spaces. One way to consider the lower bound on the Ricci curvature consists in observing that, in harmonic coordinates, the Ricci tensor can be written as the opposite of the Laplacian of the metric, plus some quadratic error terms which depend on the metric and its derivatives. Therefore, a lower bound on the Ricci curvature corresponds, in analytic terms, to an upper bound on the Laplacian of the metric. As a consequence, it is natural to replace the Laplacian by a Schrödinger operator, that is, the Laplacian minus an appropriate potential containing enough information on the curvature. In [CMT24, CMT22, CMT23b, CMT23a] we used the negative part of the Ricci curvature as a potential and assumed that it satisfies a condition inspired by Kato potentials in $\mathbb{R}^n$, as introduced by T. Kato and studied by B. Simon [Kat72, Sim82]. In this way, we obtained a regularity theory that recovers, under a much weaker assumption, the celebrated Cheeger-Colding theory for limits of manifolds with a lower Ricci bound: I present its main lines in this memoir.

Chapter 1

The Yamabe problem on stratified spaces

This chapter is devoted the presentation of the results concerning the Yamabe problem on compact stratified spaces that we have obtained in [Mon18] and in collaboration with K. Akutagawa in [AM22].

1.1 Introduction

The Yamabe problem on a smooth compact manifold (M^n, g) of dimension $n \geq 3$ consists in finding a metric of constant scalar curvature in the conformal class of g, that is,

$$[g] = \{\tilde{g} = u^{\frac{4}{n-2}} g,\, u \in \mathcal{C}^\infty(M)\}.$$

Since the work of H. Yamabe in the 1960s, it is known that if a metric in the conformal class is a minimizer for the Hilbert-Einstein functional

$$\mathcal{E} : g \mapsto \frac{a_n \displaystyle\int_M S_g \,\mathrm{d}v_g}{\mathrm{vol}_g(M)^{\frac{n-2}{n}}}, \quad a_n = \frac{n-2}{4(n-1)},$$

then it has constant scalar curvature. We refer to such a minimizer as a Yamabe metric. The existence of a Yamabe metric for any compact smooth manifold has been proven thanks to the work of N. Trudinger, T. Aubin and R. Schoen: their arguments rely on the study of a conformal invariant, the Yamabe constant

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} \mathcal{E}(\tilde{g}).$$

By using the transformation laws under conformal change for the scalar curvature and the volume, and the fact that Sobolev functions are dense in $\mathcal{C}^\infty(M)$, the Yamabe constant can also be written as follows:

$$Y(M, [g]) = \inf_{u \in W^{1,2}(M), u \neq 0} \frac{\displaystyle\int_M (|du|^2 + a_n S_g u) \,\mathrm{d}v_g}{\|u\|^2_{\frac{2n}{n-2}}}.$$

We refer to

$$Q_g(u) = \frac{\displaystyle\int_M (|du|^2 + a_n S_g u)\,\mathrm{d}v_g}{\|u\|^2_{\frac{2n}{n-2}}},$$

as the Yamabe functional. If $\tilde{g} = u^{\frac{4}{n-2}} g$ is a Yamabe metric, the function u is called a Yamabe minimizer and it solves the Yamabe equation:

$$\Delta_g u + a_n S_g u = Y_n u^{\frac{n+2}{n-2}},$$

where $Y_n = Y(\mathbb{S}^n, g_0)$ is the Yamabe constant of the round sphere $(\mathbb{S}^n, g_0)$. If the Yamabe constant $Y(M, [g])$ is non-positive, N. Trudinger [Tru68] showed that there exists a unique Yamabe metric in the conformal class $[g]$. T. Aubin [Aub76b] proved what is now generally called the "Aubin's inequality", that is, for any compact manifold (M^n, g) the Yamabe constant is smaller than or equal to Y_n; moreover, he showed that if the inequality is strict, then a Yamabe metric does exist. He then showed the strict inequality $Y(M^n, [g]) < Y_n$ in dimension $n \geq 6$ and if g is not locally conformally flat. In the remaining cases, that is, in low dimension $n = 3$, 4, 5 and for locally conformally flat metrics, R. Schoen [Sch84] used the positive mass theorem obtained with S T. Yau [SY79, SY81], in order to show that either $Y(M, [g]) < Y_n$ or the manifold is conformally equivalent to the round sphere, whose round metric is a Yamabe metric.

The study of the Yamabe problem in presence of isolated conical singularities goes back to the work of K. Akutagawa and B. Botvinnik [AB04] and of J. Viaclovsky [Via10], where they considered the case of *orbifolds*. These are topological spaces locally modelled on the quotient of the Euclidean space by an isometry subgroup. One of the easiest examples of orbifold is the so-called American football, which is the quotient of the sphere $\mathbb{S}^2$ by a group $\mathbb{Z}/q\mathbb{Z}$. It carries two isolated conical singularities of angle $2\pi/q$ at the fixed points points of $\mathbb{Z}/q\mathbb{Z}$. For an orbifold singularity modelled on $\mathbb{R}^n/\Gamma$, we refer to Γ as the local group of the singularity. A first difference with respect to the smooth case is that a generalized Aubin's inequality hold, where the Yamabe constant of the sphere is replaced by the *orbifold Yamabe constant*, which takes into account the angle of the singularities. The existence of a Yamabe metric is proven in [AB04] when the inequality is strict. However, in case of equality, a Yamabe metric does not necessarily exist. Indeed, J. Viaclovsky [Via10] constructed examples of orbifolds carrying a constant scalar curvature metric whose conformal class does not contain a Yamabe metric: these are obtained as orbifold conformal compactifications of 4-dimensional Einstein, almost locally Euclidean, hyper-Kähler manifolds.

In [ACM14, ACM15], K. Akutagawa, G. Carron and R. Mazzeo considered the more general case of stratified spaces, that carry iterated edge singularities: roughly speaking, these are singularities of codimension at least 2, modelled on the product of the Euclidean space $\mathbb{R}^j$ and a truncated cone over a link Z, which can be a compact smooth manifold or a stratified space as well. A stratified space X is composed of a regular dense set X^{reg}, an open smooth manifold of dimension n, and of singular strata Σ^j: each stratum contains the singularities of the same dimension $j \in \{0, \ldots, n-2\}$. As an example, the spherical suspension of an American football is a stratified space of dimension 3 with singularities of codimension 3 and of codimension 2: a neighbourhood of the first looks like a cone over the American football itself, while in the second case the model is the product of an interval and a cone over a circle. An iterated edge metric g on a stratified is defined by induction on the dimension of the space, and around a point in Σ^j, it is asymptotic to

$$g_{0,j} = dx^2 + dr^2 + r^2 h_j,$$

where dx^2 is the Euclidean metric on $\mathbb{R}^j$ and h_j is an iterated edge metric on the link Z_j. For the interested reader, we refer to [ACM14, ALMP12, Mon15, Mon17, BKMR21] for the precise definitions and the basic analytic tools on compact stratified spaces. In this setting, a constant scalar curvature metric has constant scalar curvature on the regular set, and a Yamabe metric in the conformal class of an iterated edge metric g is of the form $\tilde{g} = u^{\frac{4}{n-2}} g$, where u is a weak solution of the Yamabe equation and a strong solution on X^{reg}. As in the case of orbifolds, in order to prove existence of a Yamabe metric it is necessary to introduce a new conformal invariant: this was done in [ACM14], where the *local Yamabe constant* is defined. One considers the Yamabe constant of a ball $Y(B(p,r))$ by taking the infimum of the Yamabe functional Q_g over Sobolev functions with compact support on $B(p,r) \cap X^{reg}$, then defines

$$Y_\ell(X,[g]) = \inf_{p\in X} \lim_{r\to 0} Y(B(p,r)).$$

The limit in the right-hand side is equal to the Yamabe constant of the sphere Y_n for any smooth point. K. Akutagawa and B. Botvinnik showed that for an isolated orbifold singularity with group Γ, it is equal to $Y_n/|\Gamma|$, where $|\Gamma|$ is the order of the group. For a point in the stratum of dimension j and link Z, it is the Yamabe constant of the product $\mathbb{R}^j \times C(Z)$ with the metric $g_{0,j}$. The work [ACM14] showed existence of a Yamabe metric on a stratified space if the Yamabe constant is strictly smaller than the local one. However, in most cases the explicit value of the local Yamabe constant is unknown. One of the main results of my Ph.D. thesis [Mon15, Mon17] was to obtain the value of the local Yamabe constant whenever each link of the singularities carries an Einstein metric. More precisely:

Theorem 1.1. *Let (X,g) be a compact stratified space with links (Z_j,h_j), $j=1,\dots,N$ of dimension d_j. Assume that for any j, the iterated edge metric h_j is such that $\mathrm{Ric}_{h_j} = d_j - 1$ on the regular set of Z_j. Then the local Yamabe constant of X is given by*

$$Y_\ell(X,[g]) = \inf_{j=1,\dots,N} \left\{ Y_n, \left(\frac{\mathrm{vol}_{h_j}(Z_j)}{\mathrm{vol}(\mathbb{S}^{d_j})} \right)^{\frac{2}{n}} Y_n \right\}.$$

This generalized the work of K. Akutagawa and B. Botvinnik [AB04] and of J. Petean [Pet09], where the Yamabe constant of cones over a compact smooth manifold (M^n, g) with $\mathrm{Ric}_g \geq n-1$ is computed. The previous result showed in particular that in presence of one codimension 2 stratum of angle α, the local Yamabe constant is given by

$(\alpha/2\pi)^{\frac{2}{n}} Y_n$ if $\alpha \in (0, 2\pi)$ and by Y_n otherwise. This has several consequences. First, the proof in case of angle $\alpha < 2\pi$ relies on an optimal Sobolev inequality which directly implies a lower bound for the Yamabe constant, attained in the case of an Einstein metric: therefore, on a stratified space (X, g) with codimension 2 singularity of angle $\alpha < 2\pi$, if the metric g is Einstein, then it is a Yamabe metric, even in presence of higher codimension strata. Nevertheless, an Einstein metric on a compact stratified space is *not* always a Yamabe metric. Consider $\alpha > 0$, $a = \alpha/2\pi$ and the product

$$X = \left(0, \frac{\pi}{2}\right) \times \mathbb{S}^{n-2} \times \mathbb{S}^1,$$

endowed with the Einstein metric

$$h_a = \mathrm{d}\rho^2 + \sin^2(\rho) g_{\mathbb{S}^{n-2}} + a^2 \cos^2 \rho \, \mathrm{d}\theta^2.$$

The completion of X with respect to the metric h_a, denoted by $\mathbb{S}^n_a$, is a round sphere with an edge-cone singularity of angle α along a codimension 2 circle. It can also be seen as the conformal compactification of the product $\mathbb{R}^{n-2} \times C(\mathbb{S}^1_a)$. It is easy to show that the Hilbert-Einstein functional computed on h_a gives $(\alpha/2\pi)^{\frac{2}{n}} Y_n$, which is strictly larger than Y_n if $\alpha > 2\pi$. Therefore, in this latter case h_a is an Einstein metric without being a minimizer for the Einstein-Hilbert functional. However, not everything goes wrong in the case of angles larger than 2π. Let (X^n, g) be a compact stratified space of dimension larger than 6, such that g is not locally conformally flat and has one singular stratum of codimension 2 with angle $\alpha \geq 2\pi$. In this setting, Aubin's argument to show the strict inequality with respect to Y_n can be carried out identically, considering test functions supported on the regular set. Since the local Yamabe constant equals the one of the sphere, this leads to proving $Y(X[g]) < Y_\ell(X, [g])$ and therefore to the existence of a Yamabe metric.

In my subsequent work, I studied further properties of Einstein metrics on compact stratified spaces. In [Mon18], I proved a rigidity result for the first eigenvalue of the Laplacian on a compact stratified space (X, g) such that $\mathrm{Ric}_g \geq n - 1$ and the angle along the stratum of codimension 2 is smaller than 2π. This allowed me to prove that, in this setting, if g is Einstein, either it is the only Yamabe metric, up to homothety, in its conformal class, or (X, g) is isometric to a spherical suspension. Both of these rigidity statements are analogues of classical results by M. Obata [Oba62, Oba72] in the case of compact smooth manifolds.

In a collaboration with K. Akutagawa [AM22], we proved that the singular spheres $(\mathbb{S}^n_a, h_a)$ do not carry any Yamabe metric in the conformal class of h_a for any $a > 2$. This was the first singular example of non-existence of Yamabe metrics since the work of J. Viaclovsky, and the first with non-isolated singularities.

In the following, I will present these two results and the main ideas of their proof. A key tool for both is the regularity of a Yamabe minimizer u on a compact, Einstein stratified space (X, g): in presence of codimension 2 singularities of angle less than 2π, a solution to the Yamabe equation is Lipschitz and belongs to $W^{2,2}(X) \cap L^\infty(X)$; if the angles are larger than 2π, u is Hölder regular, but we still have a useful gradient estimate that allows to integrate by parts.

1.2 Einstein metrics and regularity of a Yamabe minimizer

In this section we present the regularity results for Yamabe minimizers that are needed in the following. We briefly sketch the ideas of their proof, as they also come into play in the next chapter.

The following statement was first proven in [Mon18, Lemma 4.9] when the angle of the codimension 2 stratum is smaller than 2π, then in [AM22, Proposition 3.1] for angles larger than 2π.

Proposition 1.2. *Let X^n be a compact stratified space, with singular set Σ, endowed with an Einstein metric g. Let α be the cone angle along the stratum of codimension 2. Assume that $u \in W^{1,2}(X) \cap L^\infty(X)$ is a Yamabe minimizer. Then the following holds.*

1. *If $\alpha \in (0, 2\pi]$, then u belongs to $W^{2,2}(X)$, its gradient is bounded and u is a Lipschitz function.*

2. *If $\alpha > 2\pi$, then u belongs to $C^{0,\nu}(X)$ for $\nu = 2\pi/\alpha$. Moreover for any $\varepsilon > 0$*

$$\|du\|_{L^\infty(X\setminus\Sigma^\varepsilon)} \leq C\varepsilon^{\nu-1},$$

where Σ^ε denotes the ε-tubular neighbourhood of Σ.

In both cases, the starting point of the proof is the following, that is obtained by combining Lemma 3.1 and Proposition 3.2 in [AM22].

Proposition 1.3. *Let (X, g) be a compact stratified space such that $\mathrm{Ric}_g \geq k$ for some $k \in \mathbb{R}$. Let α be the angle along the codimension 2 stratum Σ^{n-2} and $V \in L^\infty(X)$. Assume that $u \in W^{1,2}(X) \cap L^\infty(X)$ is a weak solution of*

$$\Delta_g u = Vu \tag{1.1}$$

and moreover there exists a constant $c > 0$ such that

$$\Delta_g |du| \leq c|du| \text{ on } X^{reg}. \tag{1.2}$$

Then the following hold.

(i) If $\alpha \in (0, 2\pi]$, then there exists a positive constant C such that for all $\varepsilon > 0$

$$||du||_{L^\infty(X\setminus\Sigma^\varepsilon)} \leq C\sqrt{|\ln(\varepsilon)|}. \tag{1.3}$$

(ii) If $\alpha > 2\pi$, then $u \in C^{0,\nu}(X)$ for $\nu = 2\pi/\alpha$ and there exists a positive constant C such that for all $\varepsilon > 0$

$$||du||_{L^\infty(X\setminus\Sigma^\varepsilon)} \leq C\varepsilon^{\nu-1},$$

Observe that, without the assumption on the Ricci curvature, the regularity of a solution to a Schrödinger equation $\Delta_g u = Vu$ depends on *all* of the singular strata Σ^j of the space, and in particular on the first eigenvalues of the Laplacians over the links (Z_j, h_j) (see [AM22, Proposition 3.2]). When we assume $\mathrm{Ric}_g \geq k$, we obtain for all j that $\mathrm{Ric}_{h_j} \geq \dim(Z_j) - 1$ and we can apply Lichnerowicz theorem [Mon17, Theorem 2.1] to get a simpler situation, where only the codimension 2 stratum determines the regularity of u.

The previous proposition can be applied to a Yamabe minimizer on a compact stratified space X carrying an Einstein metric g. First, since a Yamabe minimizer is bounded and Scal_g is constant, we can consider the Yamabe equation as a Schrödinger equation with the bounded potential given by

$$V = Y(X,[g])u^{\frac{4}{n-2}} - a_n \,\mathrm{Scal}_g \in L^\infty(X).$$

Also observe that the Yamabe equation can be seen as an equation of the form

$$\Delta_g u = F(u),$$

where the function F is the locally Lipschitz function given by

$$F(x) = (Y(X,[g])x^{\frac{4}{n-2}} - \mathrm{Scal}_g)x.$$

Under the assumption that g is an Einstein metric and using Bochner formula, it is not difficult to show that (1.2) is satisfied, see [Mon15, Proposition 2.3]. Therefore, the second point in Proposition 1.2 follows directly by applying Proposition 1.3.

When the angle along the codimension 2 stratum is smaller than 2π, it is possible to improve the gradient estimate (1.3) by using a *logarithmic trick*, see for instance [Mon15, Page 54]. For any $\varepsilon > 0$ we can choose a cut-off function such that $0 \leq \rho_\varepsilon \leq 1$, ρ_ε is equal to one outside of Σ^ε, vanishes on Σ^{ε^2} and it satisfies

$$\|\nabla \rho_\varepsilon\|_2 \leq \frac{C}{\sqrt{|\ln(\varepsilon)|}}$$

Moreover, using the gradient estimate (1.3), ρ_ε can be chosen such that

$$\int_{\Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2}} |\Delta_g \rho_\varepsilon||du|^2 \,\mathrm{d}v_g \leq C. \tag{1.4}$$

By Bochner-Lichnerowicz formula, we can write

$$\nabla^*\nabla du + \mathrm{Ric}_g(du) = F'(u)du, \tag{1.5}$$

then we express the Laplacian of $|du|^2$ as

$$\frac{1}{2}\Delta_g |du|^2 = (\nabla^*\nabla du, du) - |\nabla du|^2 = F'(u)|du|^2 - (n-1)|du|^2 - |\nabla du|^2 \leq C_1|du|^2 - |\nabla du|^2, \tag{1.6}$$

where in the last inequality we used that F is locally Lipschitz and u is bounded. By multiplying by ρ_ε, integrating by parts and using (1.4), we obtain

$$\int_X \rho_\varepsilon |\nabla du|^2 \, \mathrm{d}v_g \leq C_1 \int_X \rho_\varepsilon |du|^2 \, \mathrm{d}v_g + C, \tag{1.7}$$

so that the L^2-norm of $|\nabla du|$ is bounded and u belongs to $W^{2,2}(X)$. We can then prove

$$\Delta_g |du| \leq c|du| \tag{1.8}$$

weakly on X, and Moser's iteration allows to conclude that $|du|$ is bounded.

1.3 An Obata-type result for stratified spaces

In the case of a compact smooth manifold M^n of dimension $n \geq 3$ with Einstein metric g, thanks to the work of M. Obata [Oba62, Oba72] we know that the existence of a metric conformal to g, not homothetic to g and with constant scalar curvature, implies that (M, g) is isometric to the round sphere. Moreover, the Einstein-Hilbert functional on the round sphere is minimized by constant multiples of the round metric and their images under conformal diffeomorphism.

In [Mon18] we showed an analogous rigidity result for compact Einstein stratified spaces, provided that the angle along the codimension 2 stratum is smaller than 2π.

Theorem 1.4. *Let X be a compact stratified space of dimension n endowed with an Einstein metric g with cone angle $\alpha \in (0, 2\pi)$ along its stratum of codimension 2. Assume that there exists a metric $\tilde{g} \in [g]$, not homothetic to g, with constant scalar curvature. Then $\tilde{g}$ is an Einstein metric as well and (X, g) is isometric to the spherical suspension $([0, \pi] \times \hat{X}, dt^2 + \sin^2 t \, \hat{g})$ of a compact Einstein stratified space $(\hat{X}, \hat{g})$.*

The main ingredients to prove Theorem 1.4 are the regularity of a Yamabe minimizer stated above and the following rigidity result for the first eigenvalue of the Laplacian.

Theorem 1.5. *Let X be a compact stratified space of dimension n endowed with a metric g such that $\mathrm{Ric}_g \geq (n-1)$ and with cone angle $\alpha \in (0, 2\pi)$ along its stratum of codimension 2. The first eigenvalue of Δ_g is equal to n if and only if X is the spherical suspension of a compact stratified space of dimension $n-1$.*

Then the proof of Theorem 1.4 consists in showing the existence of an eigenfunction of the Laplacian associated with the eigenvalue n. In order to do this, we prove that there exists a function ϕ such that

$$\nabla d\phi = -\frac{\Delta_g \phi}{n} g, \tag{1.9}$$

by showing that the traceless Ricci tensor $E_{\tilde{g}}$ of $\tilde{g}$ vanishes (so that $\tilde{g}$ is an Einstein metric as well). We consider a cut-off function ρ_ε of the singular set Σ, vanishing on the tubular neighbourhood Σ^ε, equal to one outside of $\Sigma^{2\varepsilon}$, and study the integral

$$I_\varepsilon = \int_X \rho_\varepsilon |E_{\tilde{g}}|_g^2 dv_g = (n-2) \int_X \rho_\varepsilon (E_{\tilde{g}}, \nabla d\phi)_g dv_g,$$

where ϕ is such that $\tilde{g} = \phi^{-2}g$. The function ϕ is simply the power of a solution u to the Yamabe equation and it has the same regularity as u. When appropriately integrating by parts, we obtain the following estimate:

$$I_\varepsilon \leq c_1 \int_X \Delta_g \rho_\varepsilon |\nabla \phi|^2 dv_g + c_2 \int_X (\nabla \rho_\varepsilon, \nabla \phi)_g dv_g.$$

The regularity of ϕ is crucial. Indeed, using that $|\nabla\phi|$ is bounded, it was shown in [Mon15] that ρ_ε can be chosen so that the two summands on the right-hand side tend to zero as ε goes to zero. Therefore $|E_{\tilde{g}}|_g$ vanishes, $\tilde{g}$ is an Einstein metric and the conformal change of the traceless Ricci tensor immediately yields to equation (1.9). The existence of an eigenfunction of the Laplacian associated to n follows: for the details, we refer to the proofs of Corollaries 4.7 and 4.8 in [Mon18].

A similar situation to the one of the smooth round sphere is given by the following. Consider the singular sphere $(\mathbb{S}^n_a, h_a)$ as defined above. For $a \in (0, 1]$, the metric h_a is a Yamabe metric. Moreover, in [AM22] we proved:

Proposition 1.6. *Let $a \in (0, 1]$ and $(\mathbb{S}^n_a, h_a)$ defined as above. Assume that $h \in [h_a]$ has constant scalar curvature, then there exists a conformal diffeomorphism φ of $(\mathbb{S}^n_a, h_a)$, preserving the singular set, such that up to a constant multiple we have $h = \varphi^* h_a$. As a consequence, the Einstein-Hilbert functional is minimized in $[h_a]$ by constant multiples of h_a and their images under conformal diffeomorphism.*

In this proof we apply Theorem 1.4 to deduce that there is an isometry φ between $(\mathbb{S}^n_a, h)$ and a spherical suspension $([0, \pi] \times \hat{X}, dt^2 + \sin^2 t\, \hat{g})$ preserving the singular sets. Therefore, the so-called tangent cone at the point $p = \{0\} \times \hat{X}$, that is the cone over $\hat{X}$, is isometric to the tangent cone of $(\mathbb{S}^n_a, h)$ at $x = \varphi^{-1}(p)$. We use again that if $h = \phi h_a$, the function ϕ is Lipschitz and, as a consequence, the tangent cone of $(\mathbb{S}^n_a, h)$ at x coincides with the tangent cone of $(\mathbb{S}^n_a, h)$ at x. This latter cone is the cone over $(\mathbb{S}^{n-1}_a, h^{n-1}_a)$. This implies that $(\hat{X}, \hat{g})$ is isometric to $(\mathbb{S}^{n-1}_a, h^{n-1}_a)$: the spherical suspension of $(\mathbb{S}^{n-1}_a, h^{n-1}_a)$ is clearly $(\mathbb{S}^n_a, h_a)$, thus φ is the desired conformal diffeomorphism.

1.4 Non-existence of Yamabe metrics

As we pointed out above, for any $a > 1$, the metric h_a on $\mathbb{S}^n$ is an Einstein metric without being a Yamabe metric. Furthermore, in [AM22] we showed that for all $a \geq 2$ the conformal class of h_a does not contain any Yamabe metric. More precisely we obtained the following.

Theorem 1.7. *Let $\alpha \in [4\pi, +\infty)$ and $a = \alpha/2\pi$. Let $(\mathbb{S}^n_a, h_a)$ be defined as above. Then there is not any Yamabe metric in the conformal class of h_a.*

We briefly sketch the main ingredients of the proof and refer to [AM22, Section 5] for the details. Our proof relies on the computation of the local Yamabe constant, the regularity of a Yamabe minimizer and a contradiction argument inspired by a lemma due to Aubin [Aub76a, AN07]. This latter lemma states that the Yamabe constant of a finite covering (M_k, g_k) of a compact smooth manifold (M^n, g) is strictly greater than $Y(M,[g])$: in order to do this, one needs to consider a solution u to the Yamabe equation on (M_k, g_k), its average v over the deck transformation group of the covering and a function v_0 on (M,g) whose lift is v. Then by using integration by parts and Hölder's inequality it is not difficult to obtain $Q_g(v_0) < Y(M_k,[g_k])$. We perform this argument by considering $(\mathbb{S}_a, h_a)$ as the double branched cover of $(\mathbb{S}_b, h_b)$ for $b = a/2$. We assume by contradiction that a Yamabe metric does exists on $(\mathbb{S}_a, h_a)$ and we denote it by $\tilde{g} = u^{\frac{4}{n-2}} h_a$, where u solves the Yamabe equation on the regular set. We consider the average of u over the deck transformation group $\{\mathrm{id}, \gamma\}$

$$v = u + u \circ \gamma,$$

and define v_0 as the function on $\mathbb{S}_b^n$ whose lift is v. If we can integrate by parts and show that $Q_{h_b}(v_0) < Y_n$, we obtain a contradiction. Indeed, thanks to [Mon17] we know that $Y(\mathbb{S}_a^n,[h_a]) = Y(\mathbb{S}_b^n[h_b]) = Y_n$. The gradient estimate of Proposition 1.2 is key to allow us to be able to integrate by parts: we can write

$$\int_X |du|^2 \, \mathrm{d}v_g = \lim_{\varepsilon \to 0} \int_{X \setminus \Sigma^\varepsilon} u \Delta_g u \, \mathrm{d}v_g - \int_{\partial \Sigma^\varepsilon} u \langle du, N \rangle \, \mathrm{d}\sigma_g,$$

where N is the unit outward normal of $\partial\Sigma^\varepsilon$. Thanks to Proposition 1.2, the last term in the right-hand side is bounded by

$$\int_{\partial \Sigma^\varepsilon} u \langle du, N \rangle \, \mathrm{d}\sigma_g \leq \|u\|_\infty \varepsilon^{\nu-1} \operatorname{vol}_{\sigma_g}(\partial \Sigma^\varepsilon),$$

and since the codimension of Σ is equal to two, this volume is controlled by a constant times ε. Therefore, by passing to the limit as ε goes to zero, we obtain the classical integration by parts formula

$$\int_X |du|^2 \, \mathrm{d}v_g = \int_X u \Delta_g u \, \mathrm{d}v_g.$$

This allows us to use Aubin's computation for coverings in order to obtain the desired contradiction.

1.5 Perspectives

Many questions are still open in relation with the Yamabe problem on stratified spaces. We briefly present below the ones that we plan to address in the future.

Non-existence of Yamabe metrics

First of all, in the continuity of our previous work, the restriction $\alpha \geq 4\pi$ in Theorem 1.7 is technical: when using the double branched covering of $(\mathbb{S}_a^n, h_a)$ we need $\alpha/2 \geq 2\pi$ in order to get a contradiction. We conjecture that even in the case $\alpha > 2\pi$, the conformal class of $[h_a]$ does not contain a Yamabe metric. We intend to prove this by using a technique based on isoperimetric domains, different from the one of [AM22]. Indeed, the existence of Yamabe metrics on $(\mathbb{S}^n_a, h_a)$ is equivalent to the existence of extremal functions for the optimal Sobolev inequality in the conformal stereographic projection $\mathbb{R}^n \times C(\mathbb{S}^1_a)$. This inequality is equivalent to the Euclidean isoperimetric inequality, that does hold in $\mathbb{R}^n \times C(\mathbb{S}^1_a)$ for $a \geq 1$ as it was proven in [Mon17]. We will show that the only isoperimetric domains are the balls not intersecting the singular set. This will allow to prove that there is no extremal function for the Sobolev inequality on $\mathbb{R}^n \times C(\mathbb{S}^1_a)$, and as a consequence no Yamabe metric on $(\mathbb{S}^n_a, h_a)$ if $a > 1$. The difficulty lies in the fact that, in presence of an angle larger than 2π, the space is not of non-negative curvature, even in a generalized sense, then one cannot rely on the recent techniques of [Bre23, BK23], where isoperimetric domains are studied in manifolds and spaces of non-negative curvature.

Positive mass theorem

The positive mass theorem has been proven for dimensions between 3 and 8 [SY79, SY81] and in any dimension in the case of spin [Wit81] and Kähler [HL16] manifolds; proofs removing these assumptions have been announced by Lohkamp, and Schoen and Yau. Starting from the work of P. Miao [Mia02], there have been several recent developments in the study of positive mass theorems for low-regularity metrics, see for instance [JSZ22b, DSW24a] and the references therein. T. Ju and J. Viaclovsky [JV23] showed a positive mass theorem for asymptotically flat manifolds with finitely many isolated orbifold singularities. A common feature of these works is that the singular set is compact, thus it has no influence on the model at infinity. However, if one is interested in geometric applications to the Yamabe problem in a singular setting, these versions of the positive mass theorem do not give enough information. Indeed, in case of orbifolds or more generally (iterated) conical singularities, it is necessary to consider a conformal stereographic projection via the Green function with pole at a singular point. For an isolated orbifold singularity, the space obtained is asymptotically locally Euclidean (ALE), that is, its model at infinity is a quotient of the Euclidean space by a finite group. The mass of ALE spaces, when it is well-defined, can be negative, or vanish without the space being Euclidean, as shown by the examples of [LeB88, HL16]. If the singularities are of codimension 2, a conformal projection at a singular point would give a space with singularities at infinity. We plan to study the validity of the positive mass theorem in this setting, starting from dimension 3. Even in this case, this is a challenging question that needs to understand how the singularities affect the asymptotic expansion of the Green function and consequently the value of the mass. We intend to follow different strategies. First of all, we plan to use potential theory, taking inspiration on the recent techniques of [AMO24], that are based on a monotonicity formula for the level sets of the Green function of the Laplacian. At the same time, we intend to study minimal surfaces in the singular setting, with the goal to adapt R. Schoen and S. T. Yau's approach. Finally, a more difficult option consists in understanding the appropriate notion of spin singular manifold and the properties of the Dirac operator in this setting, in order to be able to apply Witten's techniques. In all of the three cases, the comprehension of the necessary tools in presence of singularities represents a new subject that can lead to many applications. We plan to first consider orbifold singularities along a curve, then to explore the more general case of conical singularities not necessarily obtained as quotients by finite groups. The following step will consists in studying the question of rigidity: the vanishing mass ALE examples and the case of singular spheres $(\mathbb{S}^n_a, h_a)$ show that this question will be much richer than in the smooth case.

Compactness of constant scalar curvature metrics

As N. Trudinger showed, in the case of a compact smooth manifold (M^n, g) with negative Yamabe invariant, there is a unique constant scalar curvature (CSC for short) conformal metric in $[g]$. If the Yamabe invariant is strictly positive and the manifold is not conformally equivalent to the sphere, R. Schoen conjectured that the set of CSC conformal metrics is compact. Surprisingly, this is true in dimension smaller than 24 and false otherwise, see [Bre08]. In collaboration with N. Marque and S. Tapie, we intend to prove that in presence of orbifolds singularities, the lack of compactness can occur even in dimension smaller than 24. For this, we will consider 4-orbifolds with a finite number of isolated singularities. To each singular point one can associate an ALE space with a finite number of orbifold singularities, through a conformal stereographic projection. We call mass of the singular point the mass of this ALE space: it can be positive, negative or equal to zero. The recent work [JV23] shows that if all of the masses at singular points are non-zero, then the CSC metrics in a conformal class are compact. Nevertheless, [Via10, Theorem 1.5] gives examples of existence of CSC conformal metrics with one orbifold point of vanishing mass. Therefore, in presence of points of zero mass, we plan to analytically characterize the lack of compactness, then to give geometric conditions to avoid this phenomenon, by taking inspiration on the techniques of [KL22, KL24]. When these questions will be understood, I intend to study the construction of examples of 4-orbifolds for which the CSC metrics in a conformal class are not compact.

Yamabe invariant and singular metrics

The smooth Yamabe invariant $\sigma(M)$ of a compact smooth manifold M is the supremum over all conformal classes of Riemannian metrics g on M of the Yamabe constants $Y(M, [g])$. The introduction of this invariant was motivated by H. Yamabe's attempt to find Einstein metrics on a compact manifold; it was later studied by R. Schoen and O. Kobayashi and more recently by many others, C. LeBrun, K. Akutagawa, A. Neves, J. Petean, G. Yun, B. Ammann, M. Dahl, E. Humbert, B. Botvinnik, J. Rosenberg. One main question about the Yamabe invariant is to determine its value, which is known in relatively few cases, or to find a lower bound for it. Indeed, the Yamabe constant of the sphere is an upper bound for $\sigma(M)$ for any manifold M. We refer to the introduction of [ADH13] for a nice overview on the subject. Another challenging problem consists in establishing if there exists a metric that realizes the Yamabe invariant, and whether it is an Einstein metric: this holds in the case of existence and when $\sigma(M) \leq 0$, while it is unknown for $\sigma(M) > 0$. Note that the Yamabe invariant is positive if and only if M admits a positive scalar curvature metric, and there are well-known obstructions to the existence of such metrics due to the work of M. Gromov and H. B. Lawson and of R. Schoen and S. T. Yau.

In a joint work with K. Akutagawa, we intend to prove a lower bound for the Yamabe invariant of a manifold M which admits a singular Einstein metric g with a codimension 2 singularity of angle smaller than 2π and $\mathrm{Ric}_g = \lambda g$ for $\lambda > 0$ on the regular set. Our strategy consists in showing that there exists a sequence of smooth metrics $\{g_\delta\}$ with suitable lower bound on the Ricci curvature and control of the volume, such that $Y(M, [g_\delta])$ converges to $Y(M, [g])$. This will directly lead to $\sigma(M) \geq Y(M, [g])$.

Singular Einstein Yamabe metrics may be significant to the study of the existence of a metric attaining $\sigma(M)$: the idea is to construct a sequence of singular metrics $\{g_i\}_{i\in I}$ such that $Y(M, [g_i])$ converges to $\sigma(M)$ and then study the limit of (M, g_i). An analogue approach was used by X. X. Chen, S. Donaldson and S. Sun in order to prove the existence of a Kähler-Einstein metric on a Fano manifold, see [CDS15a, CDS15b, CDS15c]: it is first necessary to show the existence of a Kähler-Einstein metric with angles smaller than 2π along a divisor, then making the angles tend to 2π allows to obtain the desired metric on the smooth manifold. Considering codimension 2 singularities is also motivated by the case of the sphere, for which $\sigma(\mathbb{S}^n) = Y_n$ and the metrics h_a considered in the previous sections eventually give a sequence of singular Einstein metrics which converges to the round metric, such that $Y(\mathbb{S}^n_a, [h_a])$ tends to Y_n as a goes to one. The existence of a singular Einstein metric is a challenging problem worth to be investigated deeply.

Chapter 2

Geometric aspects of RCD spaces

In this chapter we present the main results obtained in collaboration with J. Bertrand, C. Ketterer and T. Richard in [BKMR21] and with A. Mondino and R. Perales in [MMP22].

2.1 Introduction

The study of generalized notions of curvature bounds started with the introduction of Alexandrov spaces in the 1950s: these are metric spaces for which a lower, or an upper curvature bound, is given in terms of triangle comparison. They generalize manifolds with *sectional* curvature bounded from below or above. As for the Ricci curvature, it is a well-known fact that it is related to *volumes*: for instance, on a smooth Riemannian manifold (M^n, g), Ric_g controls the deformation of the volume of balls along a geodesic, since it appears in the expansion of the Riemannian volume measure v_g. Moreover, a lower bound K on Ric_g implies that the volume ratio at a point x between the volume v_g of a ball of radius r and the volume of a ball of the same radius in the space of constant curvature K is non-increasing (Bishop-Gromov inequality). The first to study non-smooth spaces with generalized Ricci curvature bound were J. Cheeger and T. H. Colding [CC97, CC00a]: they developed a regularity theory for Ricci limits, that is Gromov-Hausdorff limits of sequences of manifolds satisfying a Ricci lower bound. Ricci limits are endowed with limit measures and many properties of their structure depend on them. When trying to generalize the notion of a Ricci lower bound, it is then natural to consider metric measure spaces. At the beginning of the 2000s, K. T. Sturm [Stu23, Stu06] and J. Lott together with C. Villani [LV07, LV09] introduced the so-called *curvature-dimension condition* $\mathrm{CD}(K, N)$, based on optimal transport of measures, and which extends the notions of Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded above by $N \geq 1$. This led to the development of a vast theory putting into play analytic, geometric and probabilistic methods, pushing forward the study of non-smooth spaces and with consequences on smooth manifolds as well. A comprehensive overview of the subject is out of the scopes of this memoir: we refer to [Vil09, Led09, Vil19, Gig23] for some rich surveys, and in the following we focus on some geometric aspect of this theory.

A smooth Riemannian manifold, endowed with its natural metric and measure, satisfies the $\mathrm{CD}(K, N)$ condition if and only if its Ricci curvature is bounded from below by K and its dimension is smaller than N. However, Finsler manifolds can also satisfy the CD condition: it was with the goal of ruling out Finslerian structures that L. Ambrosio, N. Gigli and G. Savaré [AGS14, Gig15] refined the previous condition and introduced RCD spaces. The class of $\mathrm{RCD}(K, N)$ spaces is closed under pointed measured Gromov-Hausdorff convergence and natural geometric operations such as quotients [GGKMS18], conical and warped product constructions [Ket15, Ket13]. For several years, the main examples of RCD spaces have been manifolds with Ricci curvature bounded from below, Ricci limits, weighted manifolds with Bakry-Émery curvature bounded from below and the above cited constructions over such manifolds or RCD spaces. The singularities of these spaces are quite rigid: for instance, only exact cones or orbifolds singularities, which are obtained by a quotient, were known to belong to the setting of RCD spaces, but not general conical singularities. In parallel, singular manifolds had been studied for long in Riemannian geometry, see for instance [Che79], and in most of the cases the singularities are *asymptotic* to a model, such as a cone or a cusp, but not exactly conical or cuspidal. In his 2017 Bourbaki seminar, C. Villani [Vil19] raised the question of enlarging the class of examples of RCD spaces. The joint work [BKMR21] aims to give a new class of singular examples that are RCD spaces, building a bridge between RCD theory and the theory of singular manifolds with iterated conical singularities. As an application, our result has been recently used in [DSW24b] to show the following rigidity statement: on the connected sum of a manifold M^n, which is spin or has dimension between 3 and 7, and the torus $\mathbb{T}^n$ a metric with conical singularities and non-negative scalar curvature must be flat everywhere and extends smoothly across the singular points (see the proof of [DSW24b, Theorem 1.1]).

Another challenging question concerning RCD spaces consists in understanding their geometric and topological properties. We are concerned here in particular in the properties related to the fundamental group, which can be seen as the simplest topological invariant of a manifold, and on the first Betti number. Thanks to results of Myers and Bochner, it is well-known that a lower Ricci bound on a manifold is related to the fundamental group: if (M^n, g) is such that $\mathrm{Ric}_g \geq K$ with $K > 0$, then its fundamental group $\pi_1(M)$ is finite and its first Betti number $b_1(M)$ vanishes. When $K \geq 0$, then $b_1(M) \leq n$, with equality if and only if the manifold is isometric to the flat torus.

For RCD spaces, even the existence and uniqueness of a universal cover, thus of a topologically relevant fundamental group, is not trivial. C. Sormani and G. Wei [SW04a, SW04b, SW01] developed a theory to ensure that the universal cover of a length space does exist: this applies to Ricci limit spaces and was the starting point for A. Mondino and G. Wei [MW19] to prove the existence of a universal cover for an RCD^* space. The condition CD^* was introduced by [BS10] as a "reduced" CD condition carrying a local-to-global property. A priori, the universal cover $(\widetilde{X}, \widetilde{\mathsf{d}}, \widetilde{\mu})$ of an RCD^* space (X, d, μ) is not necessarily simply connected, but one can still define the *revised fundamental group* $\overline{\pi}_1(X)$ as the group of deck transformations of its universal cover.

The *revised Betti number* $\bar{b}_1(X)$ is defined as the rank of the abelianization of $\bar{\pi}_1(X)$. In this setting, in a joint work with A. Mondino and R. Perales [MMP22], we first proved an upper bound for $\bar{b}_1(X)$, that extends a result of M. Gromov and S. Gallot for compact smooth manifolds with almost non-negative Ricci curvature. Secondly, we obtained a topological stability result for the flat torus, recovering the one proven in the smooth setting by J. Cheeger and T. H. Colding [Col97, CC97].

Very recently, J. Wang [Wan24] proved that any RCD* space (X, d, μ) is semi-locally simply connected. As a consequence, its fundamental group is simply connected and its revised fundamental group $\bar{\pi}_1(X)$ is isomorphic to the usual fundamental group $\pi_1(X)$. The revised Betti number that we used in our work [MMP22] then coincides with the usual first Betti number of a manifold.

2.2 Minimal background on RCD spaces

Throughout this chapter a metric measure space is a triple (X, d, μ) such that (X, d) is a complete and separable metric space and μ is a locally finite Borel measure on X with full support. In the following and for the scopes of this memoir, we are going to focus on the definition of RCD metric measure spaces and on their basic properties, while we refer to the ample literature on synthetic curvature bounds, for instance to the surveys [Vil09, Gig23] for the precise definition of the $\mathrm{CD}(K, N)$ condition. This is defined by a convexity property for an entropy functional along geodesics in the space of probability measures on (X, d, μ), using optimal transport of measures.

For any $N \geq 1$ and $K \in \mathbb{R}$, L. Ambrosio, N. Gigli and G. Savaré [Gig15, AGS14] defined an $\mathrm{RCD}(K, N)$ space as a space that satisfies the $\mathrm{CD}(K, N)$ condition and which is in addition *infinitesimally Hilbertian*. In order to explain this property, we need to introduce some notations. We denote by $\mathrm{Lip}(X)$ the set of Lipschitz functions on X, and for $f \in \mathrm{Lip}(X)$ we define its local Lipschitz constant $\mathrm{Lip}(f)$ as the function

$$x \mapsto \mathrm{Lip}(f)(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}.$$

The Cheeger energy $\mathsf{Ch} : L^2(X, \mu) \to [0, +\infty]$ is the convex and lower semi-continuous functional defined for any $f \in L^2(X, \mu)$ by

$$\mathsf{Ch}(f) = \inf_{f_n \to f} \left\{ \liminf_{n \to \infty} \int_X \mathrm{Lip}(f_n)^2 \, \mathrm{d}\mu \right\}, \tag{2.1}$$

where the infimum is taken over the set of sequences $(f_n)_n \subset L^2(X, \mu) \cap \mathrm{Lip}(X)$ converging to f in $L^2(X, \mu)$. The Sobolev space of (X, d, μ) is then defined by

$$W^{1,2}(X, \mathsf{d}, \mu) = \mathcal{D}(\mathsf{Ch}) = \{f \in L^2(X, \mu),\, \mathsf{Ch}(f) < +\infty\},$$

and endowed with the norm

$$\|f\|_{W^{1,2}}^2 = \|f\|_2^2 + \mathsf{Ch}(f).$$

It is possible to show that for any $f \in W^{1,2}(X, \mathsf{d}, \mu)$ there exists a unique L^2-function $|df|$ called *minimal relaxed slope* such that

$$\mathsf{Ch}(f) = \int_X |df|^2 \, \mathrm{d}\mu,$$

and $|df| = |dg|$ for any g such that $f = g$ μ-a.e.

Definition 2.1. A metric measure space (X, d, μ) is *infinitesimally Hilbertian* if its Cheeger energy is quadratic, or equivalently, if the space $(W^{1,2}(X, \mathsf{d}, \mu), \| \cdot \|_{W^{1,2}})$ is a Hilbert space.

A Finsler manifold can satisfy the CD condition, but its Cheeger energy is quadratic if and only if the manifold is Riemannian. This was one of the main reasons to introduce the following definition.

Definition 2.2. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space (X, d, μ) is an $\mathrm{RCD}(K, N)$ space if it satisfies the $\mathrm{CD}(K, N)$ condition and it is infinitesimally Hilbertian.

Remark 2.1. We point out that one may also consider the so-called CD^* condition, which is a priori weaker than than CD, and introduce RCD^* spaces similarly. One important difference in the definitions of the CD and CD^* conditions is that this latter has the local-to-global property: if (X, d, μ) is such that for any point in x there exist a ball $B^X(x, r)$, a $\mathrm{CD}^*(K, N)$ metric measure space (Y, d_Y, μ_Y) and a measure preserving isometry $\varphi_x : B^X(x, r) \to B^Y(y, r)$, then (X, d, μ) is a $\mathrm{CD}^*(K, N)$ space. We refer to [BS10] and [EKS15, Section 3]. This property was used in the work of A. Mondino and G. Wei [MW19] to construct a universal cover of an RCD^* space that still satisfies the RCD^* condition. However, F. Cavalletti and E. Milman [CM21] have shown that the CD and CD^* conditions are equivalent for an *essentially non-branching* metric measure space of finite measure; this was later extended by Z. Li [Li24] in the case of infinite measure. Moreover, RCD^* spaces have been shown to be essentially non-branching in [RS14] (and more recently non-branching in [Den21]), therefore being RCD^* is equivalent to being RCD. For this reason, in the rest of this presentation we only refer to RCD spaces.

In this chapter and the following, it will be useful to adopt another point of view based on *Dirichlet spaces* and the Bakry-Émery inequality, rather than on optimal transport. The Bakry-Émery inequality is a weak version of the Bochner inequality that was introduced in the work of D. Bakry and M. Émery [BE85, Bak94] using the so-called Γ-*calculus*. We refer to [CMT24, Section 1.2] for the precise definition of a regular, strongly local Dirichlet space, and we recall here the notions of *carré du champ* and of self-adjoint operator associated to a Dirichlet form.

Definition 2.3. Let $(X, d, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space. The *carré du champ* is a non-negative definite symmetric bilinear map $\Gamma : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathrm{Rad}$, where Rad is the set of signed Radon measure on (X, d), such that for all $f, g \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}(f, g) = \int \mathrm{d}\Gamma(f, g).$$

A Riemannian manifold (M^n, g) is clearly a Dirichlet space when endowed with the Dirichlet form defined for any $u, v \in \mathcal{C}_c^1(M)$

$$\mathcal{E}(u,v) = \int_M \langle du, dv\rangle_g \, \mathrm{d}\mu_g, \tag{2.2}$$

where μ_g is any constant multiple of the Riemannian volume v_g. The Γ operator simply associates to any $f, g \in W^{1,2}(M)$ the measure $\langle df, dg\rangle_g \, \mathrm{d}\mu_g$.

For an infinitesimal Hilbertian space, the Cheeger energy is a strongly local, regular Dirichlet form. Moreover, the carré du champ and the minimal relaxed slope are related by the following: the *carré du champ* operator takes values in the set of absolutely continuous Radon measures, and for any $f, g \in W^{1,2}(X, \mathsf{d}, \mu)$ we can define

$$\langle df, dg\rangle := \frac{\mathrm{d}\Gamma(f,g)}{\mathrm{d}\mu} = \lim_{\varepsilon\to 0} \frac{|d(f+\varepsilon g)|^2 - |df|^2}{2\varepsilon} \in L^2(X,\mu).$$

In particular for all $f \in W^{1,2}(X, \mathsf{d}, \mu)$ we have $\mathrm{d}\Gamma(f) = |df|^2 \, \mathrm{d}\mu$.

A Dirichlet form is associated to a non-negative definite self-adjoint operator L with dense domain $\mathcal{D}(L) \subset L^2(X, \mu)$, defined by

$$\mathcal{D}(L) := \left\{ f \in \mathcal{D}(\mathcal{E}) \,:\, \exists h =: Lf \in L^2(X,\mu) \text{ s.t. } \mathcal{E}(f,g) = \int_X hg \, \mathrm{d}\mu \;\; \forall g \in \mathcal{D}(\mathcal{E}) \right\}.$$

In the case of an infinitesimally Hilbertian space this operator is referred to as the *Laplacian* and denoted by Δ. Thanks to the work of M. Erbar, K Kuwada, K. T. Sturm [EKS15], L. Ambrosio, A. Mondino and G. Savaré [AMS19], and to the equivalence between RCD and RCD*, Definition 2.2 is equivalent to the following.

Definition 2.4. Let $N \in [1, \infty)$, $K \in \mathbb{R}$. A metric measure space (X, d, μ) is an $\mathrm{RCD}(K, N)$ space if the following hold.

1. (X, d, μ) is infinitesimally Hilbertian.

2. There exists $x \in X$ and $c > 1$ such that for any $r > 0$ we have

$$\mu(B(x,r)) \le c e^{cr^2}.$$

3. The Sobolev-to-Lipschitz property is satisfied, that is, for all $f \in W^{1,2}(X, \mathsf{d}, \mu)$ such that $|df|^2 \le 1$ μ-a.e. there exists a Lipschitz representative of f.

4. The Bakry-Émery inequality BE(K,N) holds, that is, for all $f \in \mathcal{D}(\Delta)$ with $\Delta f \in W^{1,2}(X, \mathsf{d}, \mu)$ and for all $\psi \in \mathcal{D}(\Delta) \cap L^\infty(X)$, with $\psi \ge 0$ and $\Delta\psi \in L^\infty(X)$ we have

$$-\frac{1}{2}\int_X |df|^2 \Delta\psi + \psi\langle d\Delta f, df\rangle \, \mathrm{d}\mu \ge \int_X \psi \left(K|df|^2 + \frac{(\Delta f)^2}{N} \right) \mathrm{d}\mu. \tag{BE(K,N)}$$

The following are the properties of CD and RCD spaces that are relevant for the rest of this chapter and the next.

1. The Bishop-Gromov inequality holds on an CD space, see [Stu06].

2. The set of RCD metric measure spaces is compact with respect to the pointed measured Gromov-Hausdorff topology: any sequence of RCD (respectively RCD*) spaces admits a subsequence converging to a limit RCD (respectively RCD*) space. This was first proven for CD spaces in [LV09, Stu23], then obtained in [EKS15, AGS14, GMS15] in the RCD case.

3. We have the following scaling properties: if (X, d, μ) is an RCD(K, N) space, then for any $\lambda, c > 0$ the space $(X, \mathsf{d}, c\mu)$ is RCD(K, N) and $(X, \lambda\mathsf{d}, \mu)$ is RCD$(\lambda^{-2}K, N)$.

4. For any $K \in \mathbb{R}$ and $N \in (1, \infty)$ and for any RCD(K, N) space (X, d, μ) there is a unique integer $k \in \{1, \dots, \lfloor N \rfloor\}$ such that (X, d, μ) is k-rectifiable, that is, there exits a countable collection of Borel subsets $(V_i)_{i\in I}$ such that $\mu(X \setminus \bigcup_{i\in I}) = 0$ and bi-Lipschitz maps $\phi_i : V_i \to \phi_i(V_i) \subset \mathbb{R}^k$ with $\phi_i(V_i)$ Borel sets and such that $(\phi_i)_\#(\mu \llcorner V_i) \ll \mathcal{H}^k$. The integer k is called the *essential dimension* of (X, d, μ). The existence of an essential dimension was shown by [BS20], while rectifiability was proven in [MN19, KM18, GP21, BPS21].

2.3 RCD stratified spaces

In this section we present the main result of [BKMR21], where we give a criterion for a compact stratified space to be an RCD space. We refer to [BKMR21, Section 1] for the details on the structure of a compact stratified space (X^n, g) as a metric measure space. In particular, we point out that a compact stratified space (X^n, g) endowed with its natural distance d_g and measure v_g has finite volume, satisfies the Sobolev-to-Lipschitz property and is infinitesimally Hilbertian (see [BKMR21, Section 2]). Therefore, following Definition 2.4, a stratified spaces is RCD(K, N) whenever the Bakry-Émery inequality BE(K, N) is satisfied.

We had proven several geometric and analytic results on stratified spaces whose Ricci tensor is bounded from below on the regular set and for which the angle along the codimension 2 stratum is smaller than 2π: a Myers diameter bound, an optimal Sobolev inequality with explicit coefficients, a lower bound for the first eigenvalue of the Laplacian, an Obata-type rigidity statement in the case of equality in this latter bound [Mon17, Mon18]. All of these results have their counterparts in the smooth and RCD settings. This suggests that the Bakry-Émery condition should hold whenever the Ricci tensor is bounded from below on the regular set and the angles along the codimension 2 stratum are smaller than 2π. We actually proved that this assumption on the Ricci tensor and on the cone angle is a necessary and sufficient condition for a compact stratified space to be RCD.

Theorem 2.2. *Let (X^n, g) be a compact stratified space endowed with an iterated edge metric. Equipped with its natural distance d_g and measure v_g, the space (X, d_g, v_g) is* $\mathrm{RCD}(K, N)$ *if and only if $n \leq N$, on the regular set $\mathrm{Ric}_g \geq K$ and the angles along the codimension 2 stratum are smaller than 2π.*

The fact that we do not need any assumption on higher codimension strata depends on the lower bound on the Ricci curvature on the regular set, that immediately implies a Ricci lower bound on the cone sections of these strata. As for the codimension 2 stratum, which is of the form $\mathbb{R}^{n-2} \times C(\mathbb{S}^1)$, it is natural to assume that the angle is smaller than 2π in order to have a generalized notion of curvature bounded from below. Indeed, it is easy to show that a 2-dimensional cone of angle $\alpha \in (0, 2\pi]$ is an Alexandrov space of non-negative curvature; if α is larger than 2π, the cone does not admit a curvature lower bound in the sense of Alexandrov. Moreover, K. Bacher and K. T. Sturm [BS14] showed that in this case the cone cannot satisfy the CD condition.

The classical proof showing that a smooth Riemannian manifold (M^n, g) is an $\mathrm{RCD}(K, N)$ space if and only if $\mathrm{Ric}_g \geq K$ and $n \leq N$ relies on the behaviour of minimizing geodesics and thus does not immediately apply to stratified spaces. Indeed, in the case of stratified spaces, little is known about the behaviour of geodesics: for instance, minimizing geodesics between regular points may not avoid the singular set. We prove as a consequence of the RCD condition that, under the assumptions of Theorem 2.2, the regular set is *almost everywhere convex*, see Proposition 4.7 in [BKMR21].

As for the proof of Theorem 2.2, showing that the $\mathrm{RCD}(K, N)$ condition on a stratified space (X^n, g) implies that $\mathrm{Ric}_g \geq K$ on the regular set relies on classical arguments in the theory of CD spaces. In order to get that the angle along the codimension 2 stratum is smaller than 2π, we use the structure of tangent cones of a stratified space and the stability property of RCD spaces with respect to Gromov-Hausdorff convergence: we refer to [BKMR21, Section 5.1] for the details.

The proof of the other implication consists in proving the Bakry-Émery inequality $\mathrm{BE}(K, N)$ under our assumptions. Since eigenfunctions of the Laplacian form a basis for its domain, we first obtain $\mathrm{BE}(K, N)$ for eigenfunctions. The Bochner inequality

$$-\frac{1}{2}\Delta_g |\nabla u|^2 + \langle \nabla \Delta_g u, \nabla u \rangle_g \geq \frac{(\Delta_g u)^2}{n} + K|\nabla u|^2. \tag{2.3}$$

holds on the regular set X^{reg}, so to prove $\mathrm{BE}(K, N)$, we would like to apply the previous inequality to an eigenfunction, multiply by a cut-off function ρ_ε of Σ and make ε go to zero. When doing this, it is necessary to deal with terms of the form

$$\int_{X \setminus \Sigma^\varepsilon} \langle \nabla \varphi, \nabla \rho_\varepsilon \rangle_g \, \mathrm{d}v_g, \quad \int_{X \setminus \Sigma^\varepsilon} \Delta \rho_\varepsilon |\nabla u|^2 \, \mathrm{d}v_g. \tag{2.4}$$

In order to make these terms go to zero, we need to get information on the regularity of eigenfunctions and to carefully choose the cut-off functions. In a similar way to the one illustrated in Section 1.2 we obtain:

Proposition 2.3. *Let (X, g) be a compact stratified space such that $\mathrm{Ric}_g \geq k$ and the angle along the codimension 2 stratum is smaller than 2π. Let $\varphi \in W^{1,2}(X) \cap L^\infty(X)$*

be an eigenfunction of the Laplacian Δ_g*. Then* φ *belongs to* $W^{2,2}(X)$ *and it is Lipschitz with bounded gradient.*

For an eigenfunction φ, the argument presented in Section 1.2 applies directly, and it does not require g to be an Einstein metric, as this assumption was only used to have a bounded potential V in our rewriting of the Yamabe equation as a Schrödinger equation of the form $\Delta_g u = Vu$.

With this result in hand, we can construct a family of cut-off functions ρ_ε such that $\|\nabla\rho_\varepsilon\|_2$ and $\|\Delta_g\rho_\varepsilon\|_1$ tend to zero as ε goes to zero. This ensures that for eigenfunctions and their finite linear combinations the Bakry-Émery inequality BE(K,N) does hold.

In order to obtain BE(K,N) for all functions f in $\mathcal{D}(\Delta_g)\cap L^\infty(X)$ with $\Delta_g f \in W^{1,2}(X)$, there is one last subtlety. We prove, see [BKMR21, Proposition 5.7]:

Proposition 2.4. *Let* (X,g) *be a compact stratified space such that* $\mathrm{Ric}_g \geq k$ *and the angle along the codimension 2 stratum is smaller than* 2π*. Then BE*(K,N) *holds for any* f *and* ψ *satisfying the following:* f *belongs to* $\mathcal{D}(\Delta_g)$ *and* $\Delta_g f \in W^{1,2}(X)$*;* ψ *belongs to* $\mathcal{D}(\Delta_g)\cap L^\infty(X)$ *and is such that* $\psi \geq 0$*,* $|\nabla\psi|$ *and* $\Delta_g\psi$ *are bounded.*

In order to obtain BE(K,N), we need to drop the additional assumption of bounded gradient for the test function. In order to do this, for any non-negative bounded ϕ in $\mathcal{D}(\Delta_g)$ with bounded Laplacian, we consider a sequence of linear combinations of eigenfunctions $\{\psi_i\}_i$ that converges to ψ in $W^{1,2}(X)$, with $\Delta_g\psi_i$ converging in $L^2(X)$ to $\Delta_g\psi$. Then we use the heat semi-group $\{P_t\}_{t>0}$ of the Laplacian: for any $t>0$, $P_t\psi_i$, $|\nabla P_t\psi_i|$ and $|\Delta_g P_t\psi_i|$ are all bounded. Then we can apply Proposition 2.4 to any f and $P_t\psi_i$ for all $t>0$ and i. A Poincaré inequality holds on a compact stratified space, so that $\{P_t\}_{t>0}$ is ultra-contractive from $L^1(X)$ to $L^\infty(X)$: this implies that $P_t\psi_i$ converges uniformly to $P_t\psi$. Finally, by making t go to zero, we obtain BE(K,N) for any f and test function ψ as in Definition 2.4.

2.4 A torus stability result for RCD spaces

As we recalled in the introduction, a classical theorem by Bochner states that if (M^n,g) is a compact manifold such that $\mathrm{Ric}_g \geq 0$, then its first Betti number $b_1(M)$ not larger than the dimension n, with equality if and only if the manifold is isometric to the flat torus. M. Gromov [Gro81] and S. Gallot [Gal83] improved this by showing that the upper bound on the first Betti number still holds if the Ricci curvature is *almost non-negative*.

Theorem 2.5 ([Gro81, Gal83])**.** *Let* $n\in\mathbb{N}$*. There exists* $\varepsilon>0$ *only depending on* n *such that if* (M^n,g) *is a compact manifold of diameter* D *satisfying* $D^2\,\mathrm{Ric}_g \geq -\varepsilon$*, then* $b_1(M)\leq n$*.*

Therefore, a natural question is: what happens if (M^n,g) has almost non-negative Ricci curvature and first Betti number equal to n? The answer was given by T. H. Colding [Col97] and J. Cheeger and T. H. Colding [CC00a] and reads as follows.

Theorem 2.6 ([Col97, CC00a]). *Let $n \in \mathbb{N}$. There exists ε only depending on n such that if (M^n, g) is a compact manifold of diameter D and such that $b_1(M) = n$ and $D^2 \operatorname{Ric}_g \geq -\varepsilon$, then (M^n, g) is diffeomorphic to a flat torus $\mathbb{T}^n$.*

The proof of this theorem relies on the study of Ricci limits: first, Colding proved that under the above assumption, large balls in the appropriate covering of M are Gromov-Hausdorff close to balls in the Euclidean space $\mathbb{R}^n$. He used equivariant Gromov-Hausdorff convergence, as introduced by K. Fukaya and T. Tamaguchi [Fuk86, FY92] to pass this information to the quotient: he obtained that (M^n, g) and the flat torus are homeomorphic for $n \neq 3$. The diffeomorphism follows from a deep result of [CC97, Theorem A.1.12]: two smooth manifolds with a lower Ricci bound that are Gromov-Hausdorff close are diffeomorphic. This is a consequence of J. Cheeger and T. H. Colding's intrinsic Reifenberg theorem.

In a joint work with A. Mondino and R. Perales [MMP22], we generalized to RCD^* spaces the two previous theorems. Thanks to the equivalence of RCD and RCD^* spaces explained in Remark 2.1, we can state our results for RCD spaces. We also point out that the recent result of J. Wang [Wan24] ensures that the universal covering of an RCD^* space is simply connected and therefore the so-called *revised fundamental group* defined by A. Mondino and G. Wei, that is, the group of deck transformations of the universal cover, is isomorphic to the fundamental group. As a consequence, the results that we stated in [MMP22] can be all formulated replacing the revised fundamental group by the usual fundamental group. Recall that the *first Betti number* can be defined as follows: let (X, d, μ) be a compact $\mathrm{RCD}^*(K, N)$ space for $K \in \mathbb{R}$ and $N \in (1, \infty)$ and $\pi_1(X)$ its fundamental group. Set $H = [\pi_1(X), \pi_1(X)]$ its commutator and $\Gamma = \pi_1(X)/H$. We know that $\pi_1(X)$ is finitely generated (see Proposition 2.25 in [MMP22]), thus Γ is also finitely generated and Abelian. As a consequence, there exists $s, s_1, \ldots, s_\ell \in \mathbb{N}$ and prime numbers $p_1, \ldots, p_\ell$ such that $\Gamma = \mathbb{Z}^s \times Z_{p_1}^{s_1} \times \ldots \times \mathbb{Z}_{p_\ell}^{s_\ell}$. The first Betti number is defined by $b_1(X) = s$.

We proved the following upper bound for $b_1(X) = s$.

Theorem 2.7. *There exists a positive function $C(N, t) > 0$ with $\lim_{t \to 0} C(N, t) = \lfloor N \rfloor$ such that for any compact $\mathrm{RCD}(K, N)$ space (X, d, μ) with $\operatorname{supp}(\mu) = X$, $\operatorname{diam}(X) \leq D$, for some $K \in \mathbb{R}, N \in [1, \infty), D > 0$, the first Betti number satisfies $b_1(X) \leq C(N, KD^2)$. In particular, for any $N \in [1, \infty)$ there exists $\varepsilon(N) > 0$ such that if (X, d, μ) is a compact $\mathrm{RCD}(K, N)$ space with $\operatorname{diam}(X) \leq D$, $KD^2 \geq -\varepsilon(N)$ then $b_1(X) \leq \lfloor N \rfloor$.*

The proof of this theorem relies on two main ingredients: a generalization of a lemma by Gromov and the Bishop-Gromov inequality applied to the Abelian covering $\overline{X} = \widetilde{X}/H$, with H defined as above.

As for the first, we proved it for compact geodesic spaces which admit a universal cover and such that Γ is finitely generated: we obtained that Γ admits a finite-index subgroup Γ', isomorphic to $\mathbb{Z}^{b_1(X)}$, such that the action of Γ' sends a point $\bar{x}$ at a distance bounded below and above in terms of the diameter of X (see Lemma 3.2 in [MMP22]).

As for the Bishop-Gromov inequality, G. Wei and A. Mondino showed that the universal cover $\widetilde{X}$ of an RCD^* space (X, d, μ) endowed with its natural distance $\widetilde{\mathsf{d}}$ and measure $\widetilde{\mu}$) is an RCD* space as well. We need quotients of the universal covering by normal subgroups of $\pi_1(X)$ to be RCD* too: we proved this in [MMP22, Corollary 2.26]. With these results in hand, a contradiction argument and volume counting argument lead to Theorem 2.7.

In order to obtain torus stability, we need to keep into account that the upper bound N on the dimension is not necessarily an integer, and the essential dimension of the space may be strictly smaller than $\lfloor N \rfloor$, in which case we would not get an isomorphism with the flat torus of dimension $\lfloor N \rfloor$. However, even if N is not an integer, we are able to show that the essential dimension of the space is equal to $\lfloor N \rfloor$ and that the finite covering $\overline{X}/\Gamma'$ is isomorphic to the flat torus $\mathbb{T}^{\lfloor N \rfloor}$. When N is integer, we obtain the analogue of Theorem 2.6. More precisely, we have the following result.

Theorem 2.8 (Torus stability for RCD(K, N) spaces)**.** *For every $N \in [1, \infty)$ there exists $\delta(N) > 0$ with the following property. Let (X, d, μ) be a compact* RCD(K, N) *space with $K \mathrm{diam}(X)^2 > -\delta(N)$ and $b_1(X) = \lfloor N \rfloor$.*

1. *The essential dimension of (X, d, μ) is equal to $\lfloor N \rfloor$. If in addition $N \in \mathbb{N}$, then $\mu = c\mathcal{H}^N$ for some constant $c > 0$.*
2. *There exists a finite cover $(X', \mathsf{d}_{X'}, \mu_{X'})$ of (X, d, μ) and a real valued function of δ and N satisfying that $\lim_{\delta \to 0} \varepsilon(\delta|N) = 0$, for every fixed N such that*

$$\mathsf{d}_{\mathrm{GH}}((X', \mathsf{d}_{X'}), (\mathbb{T}^{\lfloor N \rfloor}, \mathsf{d}_{\mathbb{T}^{\lfloor N \rfloor}}) \le \varepsilon(\delta|N),$$

where d_{GH} is the distance associated to the Gromov-Hausdorff topology and $\mathbb{T}^{\lfloor N \rfloor}$ is the flat torus.

3. *If in addition $N \in \mathbb{N}$, then (X, d) is bi-Hölder homeomorphic to an N-dimensional flat torus.*

We are going to illustrate the ideas of the proof of Theorem 2.8, as they are contained in Sections 4, 5 and 6 of [MMP22]. The key point to obtain Theorem 2.8 consists in showing that the Abelian covering $\overline{X}$ is locally mGH-close to the Euclidean space of dimension $n = \lfloor N \rfloor$. More precisely we prove:

Proposition 2.9. *Let $N \in (1, \infty)$, $\varepsilon \in (0, 1)$. There exists $\delta(\varepsilon, N)$ such that for any $\delta \in (0, \delta(\varepsilon, N)]$ if (X, d, μ) is an* RCD$(-\delta, N)$ *space with*

$$b_1(X) = \lfloor N \rfloor, \quad \mathrm{diam}(X) = 1,$$

then there exists $\overline{x} \in \overline{X}$ such that

$$\mathsf{d}_{\mathrm{mGH}}(B^{\overline{X}}(\bar{x}, \varepsilon^{-1}), \mathbb{B}^{\lfloor N \rfloor}(0, \varepsilon^{-1})) < \varepsilon.$$

An analogue result was also the starting point of Colding's original proof. As in T. H. Colding's case, we prove Proposition 2.9 by an induction argument. However, there are some substantial differences: T. H. Colding constructed inductively n points

$p_1, \ldots, p_n$ in the Abelian covering at large distance and such that the gradients of the distance functions $\mathsf{d}(\cdot, p_i)$ are almost orthogonal. For this, he used harmonic smoothings of the distance functions, the so-called almost splitting maps $\phi : \overline{X} \to \mathbb{R}^n$ with controlled average of $\langle \nabla\phi_i, \nabla\phi_j \rangle$ and Hessian (see Section 3.6.4 and Definition 3.9 in the next chapter). Even if almost splitting maps are defined for RCD spaces, we did not follow this path. We directly constructed mGH-isometries instead, relying on an almost splitting theorem by A. Mondino and A. Naber [MN19] and a volume counting argument. In particular, we only needed to control the gradient of excess functions and not their Hessian. We briefly explain below our inductive argument.

The basis of induction is given by the following result, see Proposition 4.2 and Corollary 4.3 in [MMP22].

Proposition 2.10. *For any* $\varepsilon \in (0,1)$ *there exists* $\delta_1(\varepsilon, N)$ *such that for any* $\delta \in (0, \delta_1(\varepsilon, N)]$ *and for any* $\mathrm{RCD}(-\delta, N)$ *space* (X, d, μ) *with*

$$b_1(X) = \lfloor N \rfloor, \quad \mathrm{diam}(X) = 1,$$

there exists $\bar{x}_1 \in \overline{X}$*, an* $\mathrm{RCD}(0, N-1)$ *space* $(Y_{\varepsilon,1}, \mathsf{d}_{Y_{\varepsilon,1}}, \mu_{Y_{\varepsilon,1}})$ *and* $y_{\varepsilon,1} \in Y_{\varepsilon,1}$ *such that*

$$\mathsf{d}_{\mathrm{mGH}}(B^{\overline{X}}(\bar{x}, \varepsilon^{-1}), B^{\mathbb{R}\times Y_{\varepsilon,1}}((0, y_{\varepsilon,1}), \varepsilon^{-1})) < \varepsilon.$$

The proof of the previous statement relies on the almost splitting theorem [MN19, Theorem 5.1] of A. Mondino and A. Naber. This roughly states the following. There exists δ small enough such that if (X, d, μ) is an $\mathrm{RCD}(-\delta, N)$ space, p, q are points in X at a large enough distance and their excess function

$$x \mapsto e_{p,q}(x) = \mathsf{d}(p, x) + \mathsf{d}(q, x) - \mathsf{d}(p, q),$$

and its derivative are controlled by δ, then there exists a large ball in X which almost splits a line. This means that the ball is (measured) Gromov-Hausdorff close to a large ball in a product $\mathbb{R} \times Y$ for an $\mathrm{RCD}(0, N-1)$ space (Y, d_Y, μ_Y). The almost splitting theorem clearly applies as soon as the space has large enough diameter and the estimates on the excess function hold.

In the case of the Abelian cover $\overline{X}$, a consequence of the generalized Gromov's lemma is that $\overline{X}$ has infinite diameter. We then use Abresch-Gromoll excess estimates (see [MMP22, Theorem 2.11]) and the references therein) to control the excess function. This proof does not rely on the revised first Betti number being equal to $\lfloor N \rfloor$.

As for the induction step we assume the following:

Assumption $\mathbf{A}_k$. Let $k \in \{2, \ldots, \lfloor N \rfloor - 1\}$. For any $\eta \in (0,1)$ there exists $\delta_k(\eta, N)$ such that for all $\delta \in (0, \delta_k(\eta, N)]$, if (X, d, μ) is a compact $\mathrm{RCD}(-\delta^{2\beta}, N)$ with

$$\mathrm{diam}(X) = 1, \ b_1(X) = \lfloor N \rfloor$$

there exist $\bar{x}_k \in \overline{X}$, an $\mathrm{RCD}^*(0, N-k)$ space $(Y_{\eta,k}, \mathsf{d}_{Y_{\eta,k}}, \mu_{Y_{\eta,k}})$ and $y_{\eta,k} \in Y_{\eta,k}$

$$\mathsf{d}_{mGH}(B^{\overline{X}}_{\eta^{-1}}(\bar{x}_k), B^{\mathbb{R}^k\times Y_{\eta,k}}_{\eta^{-1}}(0^k, y_{k,\eta})) < \eta.$$

Under this induction assumption, we show [MMP22, Lemma 4.4] a lower bound on the diameter of the space $Y_{k,\eta}$ that allows us to apply again the almost splitting theorem to $Y_{k,\eta}$ and to conclude the proof of Proposition 2.9.

Proposition 2.11. *If $\mathbf{A_k}$ holds, then there exist $c, \eta_0 \in (0,1)$ only depending on N such that for all $\eta \in (0,\eta_0]$ and $\delta \in (0, \delta_k(\eta, N)]$*

$$\mathrm{diam}(B^{Y_{\eta,k}}(y_{k,\eta}, \eta^{-1})) > c\eta^{-1}.$$

This uses a consequence of Gromov's generalized lemma: any ball of radius R in $\overline{X}$ contains at least $\lfloor R \rfloor^{b_1(X)}$ disjoint balls of radius $1/2$ (see [MMP22, Corollary 3.3]). The assumption $b_1(X) = \lfloor N \rfloor$, a contradiction and volume counting arguments then lead to the above diameter bound.

Once Proposition 2.9 is proven, the first claim in Theorem 2.8 is a consequence of the structure theory of RCD spaces developed in [MN19], of the constancy of the essential dimension proven by [BS20] and of a result of [Hon20b]. We first obtain that the essential dimension of the Abelian covering $\overline{X}$ is equal to $\lfloor N \rfloor$. Since the covering map is a local isomorphism of metric measure spaces, preserving the measures, X has also essential dimension $\lfloor N \rfloor$ and μ is absolutely continuous with respect to $\mathcal{H}^{\lfloor N \rfloor}$. Whenever N is integer, [Hon20b, Corollary 1.3] implies that μ is a constant multiple of $\mathcal{H}^N$.

As for the second statement, we use equivariant Gromov-Hausdorff topology as in [Col97] to show the following. Assume that $(X_i, \mathsf{d}_i, \mu_i)$ is a sequence of $\mathrm{RCD}(-\delta_i, N)$ spaces with δ_i tending to zero, $\mathrm{diam}(X_i) = 1$ and $b_1(X_i) = \lfloor N \rfloor$, with subgroups $\Gamma_i \cong \mathbb{Z}^{\lfloor N \rfloor}$ obtained as in the gerenalized Gromov's lemma. Then there are points $x_i \in X_i$ such that the sequence $(\overline{X}_i, \mathsf{d}_{\overline{X}_i}, \Gamma'_i, \overline{x}_i)$ converge in equivariant Gromov-Hausdorff topology to $(\mathbb{R}^{\lfloor N \rfloor}, \mathsf{d}_{\mathbb{R}^{\lfloor N \rfloor}}, \mathbb{Z}^{\lfloor N \rfloor}, 0)$. As a consequence, the covering $X'_i = \overline{X}_i/\Gamma'_i$ is Gromov-Hausdorff close to the flat torus $\mathbb{T}^{\lfloor N \rfloor}$.

When N is integer, the fact that $\mu = c\mathcal{H}^N$ allows us to improve this Gromov-Hausdorff convergence of metric spaces to *measured* Gromov-Hausdorff convergence of metric measure spaces. Then we can apply an analogue result to the one of [CC00a] due to V. Kapovitch and A. Mondino [KM21]: an RCD space which is measured Gromov-Hausdorff close to a smooth compact manifold is bi-Hölder homeomorphic to it. The conclusion finally follows by proving that Γ_i is torsion free, so that $X_i = X'_i$.

2.5 Perspectives

We briefly present some perspectives related to our previous work.

RCD spaces which are not limit of manifolds

It is known that the spherical suspension X over $\mathbb{RP}^2$ is a compact stratified space which is RCD without being the non-collapsed limit of Riemannian manifolds with Ricci curvature bounded from below. Indeed, M. Simon [Sim12] proved that a non-collapsed Gromov-Hausdorff limit of 3-manifolds with a lower bound on the Ricci curvature must be a topological manifold, which X is not. One may wonder if there are other examples of RCD spaces that do not arise as non-collapsed, or collapsed, limits of manifolds: a possible path could be to focus on simple RCD spaces such as manifolds with isolated conical singularities of angle less than 2π and Ricci curvature bounded from below. Even in this case, this problem is related to the difficult question of "desingularization", through Gromov-Hausdorff limits or possibly other techniques, such as Ricci flow.

Criteria for non-compact singular spaces to be RCD

Our criterion for stratified spaces to be RCD applies in the compact case: this is related mainly to the information on the regularity of the eigenfunctions of the Laplacian. It would be interesting to study the analogue situation in the complete non-compact case, possibly using harmonic functions and the heat kernel instead of eigenfunctions. Studying harmonic functions on complete stratified space may be useful in itself, for example in order to obtain information on the behaviour of minimal surfaces. The study of minimal surfaces in non-smooth spaces has been recently addressed in the work of A. Lytchak and S. Wenger [LW18b, LW18a, LW20] or of A. Mondino and D. Semola [MS23].

We also point out that S. Honda studied in [Hon18] what he called "almost smooth metric spaces", and gave two possible criteria for them to be RCD spaces, that are proven in the compact case. Similarly, Theorem 1.5 in [DWWW24] proves that a connected closed manifold of dimension $n \geq 6$ with a metric which is smooth and has a lower bound on the Ricci curvature away from a closed singular set of codimension larger than 6 is an RCD space. Any progress in the case of non-compact stratified spaces may enlighten how to obtain similar results for other kinds of non-compact singular manifolds.

First Betti number of an RCD space

As we pointed out in the introduction of this chapter, the fact that the first fundamental group of an RCD space is isomorphic to the group of deck transformations of its universal cover allows one to define the first Betti number as the rank of the abelianization of the fundamental group, as in the case of smooth manifolds. Yet, in the Riemannian setting, $b_1(M)$ is also equal to the dimension of the first de Rham cohomology group, thanks to Poincaré duality. N. Gigli and C. Rigoni [GR18] followed this approach to define the first Betti number of an RCD space and used it to prove a Bochner theorem. Whether the two possible definitions of the Betti number coincide is still an open and challenging question in RCD theory, that we would like to address. However, considering the intense activity in the study of RCD spaces in the past decade, this question might well find an answer before we have significant advances in this direction.

Chapter 3

Limits of manifolds with a Kato bound on the Ricci curvature

This chapter is devoted to presenting the main results obtained in collaboration with G. Carron and D. Tewodrose in [CMT24, CMT22, CMT23b, CMT23a].

3.1 Introduction

Since Gromov's pre-compactness theorem, limits of manifolds with a lower Ricci bound have been the object of a vast study: thanks to the work of many mathematicians, M. Anderson, Bando, Kasue, Nakajima, J. Cheeger, T. H. Colding, G. Tian, A. Naber, W. Jiang, we have now a good understanding of Ricci limits. Nevertheless, in many interesting geometric situations, such as the study of singularities of geometric flows, of variational problems or of moduli spaces of metrics, convergence of manifolds is needed without having a uniform lower Ricci bound. For instance, in [Bam17, Bam18, BZ17, Sim20a, Sim20b], the authors considered the Ricci flow with bounded scalar curvature; in [TZ16] an L^p bound on the Ricci curvature is preserved along the Kähler-Ricci flow. In the work of G. Tian and J. Viaclovsky [TV05a, TV05b, TV08] the authors studied moduli spaces of what they defined as "extremal metrics": these are metrics for which the Laplacian of the Ricci tensor is equal to a linear combination of terms depending on the Riemannian curvature and on the Ricci tensor as well. Such extremal metrics include for example Bach-flat metrics, which are critical for the Weyl functional $g \mapsto \|Weyl_g\|_2$, dual and anti-self dual metrics with constant scalar curvature, Kähler constant scalar curvature metrics.

All of these problems motivate the study of weaker conditions on the curvature, under which structure and regularity results for limit spaces can still be proved. Starting from the work of S. Gallot, P. Petersen and G. Wei [Gal88, PW97, PW01], the case of sequences of n-manifolds for which the negative part of the Ricci curvature satisfies a uniform L^p bound, for $p > n/2$, has been widely studied: we refer to [DWZ18, Che22, Ket21] for some recent results in this setting. In collaboration with G. Carron and D. Tewodrose, we considered a condition which is weaker than the L^p bound and takes inspiration from Kato potentials in $\mathbb{R}^n$, as defined by [Kat72, Sim82]. As we point out below, a potential in $L^p(\mathbb{R}^n)$ for $p > n/2$ is a Kato potential, while there are Kato potentials that do not belong to $L^p(\mathbb{R}^n)$. In order to illustrate our assumptions, we observe that a well-known fact in Riemannian geometry is that the Ricci tensor can be expressed, in harmonic coordinates, in terms of the Laplacian of the metric (see for instance [Pet06, Lemma 49]):

$$\mathrm{Ric}_g = -\frac{1}{2}\Delta g + Q(g, \partial g),$$

where Q is a quadratic error term depending on the metric and its derivatives. As a consequence, the condition $\mathrm{Ric}_g \geq K$ can be seen, roughly speaking, as $\Delta g \leq -K$: this explains why it is reasonable to expect good regularity results for a converging sequence of metrics that uniformly satisfies such an inequality. In our work together with G. Carron and D. Tewodrose, the idea consists in replacing the operator Δ by a Schrödinger operator $\Delta_g - V$, where V is the negative part of the Ricci curvature, that we denote Ric_-. We then defined appropriate Kato bounds for sequences of manifolds and developed a regularity theory for their limit spaces.

For a complete manifold (M^n, g) we defined the Kato constant $k_t(M^n, g)$ at time $t > 0$ as the L^∞ norm of the function obtained by integrating Ric_- first against the heat kernel of the Laplacian Δ_g, then over time between 0 and t. A uniform bound on $k_t(M^n, g)$ has been studied especially in the case of compact manifolds, see for instance [Car19, Ros19, CR20, RS17, RW22], and a result of G. Carron [Car19] ensures precompactness of sequences of closed manifolds $\{(M^n_\alpha, g_\alpha, o_\alpha)\}_{\alpha\in A}$ for which there exists $T > 0$ such that $k_T(M^n_\alpha, g_\alpha) \leq 1/16n$. Kato potentials on manifolds have also been studied for example in [GÏ4, GÏ7, Dev19, Dev21].

In a series of paper [CMT24, CMT22, CMT23b, CMT23a] we considered three possible bounds on sequences of complete manifolds:

- a Dynkin bound, that is a uniform bound on $k_t(M^n_\alpha, g_\alpha)$ by a constant only depending on the dimension;
- a Kato bound, in which case there exists a bounded, non-negative, non-decreasing function f tending to zero at zero and such that $k_t(M^n_\alpha, g_\alpha) \leq f(t)$ for all t and α;
- a strong Kato bound, for which the function f satisfies an additional integrability condition that we illustrate below.

A first reason to introduce the Kato bound is to study tangent cones of the limit space, that are limits of rescaled manifolds $(M^n_\alpha, \varepsilon_\alpha^{-2} g_\alpha)$ with ε_α tending to zero: under a Kato bound, the rescaling property of the Kato constant ensures that, for any fixed $t > 0$, the Kato constants at time t of the rescaled manifolds $(M^n_\alpha, \varepsilon_\alpha^{-2} g_\alpha)$ tends to zero as α goes to infinity. One can see this as an analogue of what happens for tangent cones of Ricci limits, which are approximated by rescaled manifolds of almost non negative curvature, that is, $\mathrm{Ric}_{g_\alpha} \geq K\varepsilon_\alpha$.

There are two scenarios in the study of Gromov-Hausdorff limits of smooth manifolds: the collapsing case, that is, the volume of a unit ball $B(o_\alpha, 1)$ tends to zero along the sequence; and the non-collapsing case, for which we have a uniform lower bound on the volume of $B(o_\alpha, 1)$. In our work, in both cases we obtained regularity results for limits of manifolds satisfying a Dynkin, Kato or strong Kato bound. For collapsing sequences of n-manifolds, we obtained k-rectifiability under a Dynkin bound, for some k between 1 and n. If a Kato bound holds, we proved moreover Mosco convergence of the energies and, for closed manifolds, of the spectra of the Laplacians. In the case of non-collapsing n-manifolds with a strong Kato bound, we proved that tangent cones are metrics cones, volume continuity, a stratification result and Reifenberg regularity. Our work extends the classical Cheeger-Colding theory and many of the known results in the case of L^p-bounds, under a much weaker bound. Our results have been applied in order to obtain a torus stability result in [CMT23b], or in the setting of Ricci flow by M. Lee [Lee24], who proved that a compact 3-dimensional non-collapsed strong Kato limit is homeomorphic to a smooth manifold.

In the following, we present our main results and the definitions needed to precisely state them, together with the strategy of some chosen proofs. In [CMT24, CMT22], we considered sequences of closed manifolds; the main result of [CMT23a] allowed us to improve and extend our previous work to the complete case thanks to the use of RCD theory. In this memoir, we underline the significant aspects of our work that distinguish it from the previous literature concerning convergence of manifolds. First of all, we make use of the tools for convergence of PI-Dirichlet spaces, which lead in particular to heat kernel convergence without any synthetic curvature assumption. This allows us to obtain, for example, energy convergence, or the fact that tangent cones of Kato limits are $\mathrm{RCD}(0, n)$ spaces. We also introduce new monotone quantities based on a heat kernel ratio, instead of relying on the volume density as in Cheeger-Colding theory, and use them to prove, for instance, that tangent cones of non-collapsed strong Kato limits are metric cones, or to obtain new almost rigidity results. One of the difficulties in relying on the heat kernel rather than on the volume density, is that this latter quantity is local, while the heat kernel is global. However, in some cases we are able to perform direct proofs instead of proofs by contradiction, that are extensively used in Cheeger-Colding theory: see for example the proof that tangent cones are metric cones, presented in subsection 3.6.4.

In many of our proofs in [CMT24, CMT22], a key role is played by a Li-Yau inequality that was first proven in the compact case by G. Carron in [Car19]. This restriction was dropped, thanks to the fact that our results in [CMT23a] guarantee existence of good cut-off functions on complete manifolds satisfying a Dynkin bound: this allows us to obtain the Li-Yau inequality in the complete case as well. As a consequence, the strategies of [CMT24, CMT22] apply also in the complete case: we present them here without assuming compactness. We believe that our approach, which does not need classical tools of Cheeger-Colding theory such as the almost splitting theorem or segment inequality, and which lightly uses RCD theory, could prove itself relevant to other settings.

3.2 Kato potentials in $\mathbb{R}^n$

Kato potentials were introduced by Kato in 1972 [Kat72] and then extensively studied in the setting of Euclidean space, see for instance [Sim82, AS82]. We recall here few basic facts about them.

The Kato constant of a locally integrable, non-negative function V is the L^∞-norm of the function obtained by integrating twice V, first against the heat kernel of the Laplacian and then over time. We define a Kato potential as follows.

Definition 3.1. Let $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq 0$, $t > 0$. The Kato constant of V at time t is given by

$$k_t(V) = \sup_{x \in \mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} (4\pi s)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4s}\right) V(y)\,\mathrm{d}y. \tag{3.1}$$

A potential V is said to be a Kato potential if

$$\lim_{t \to 0} k_t(V) = 0. \tag{3.2}$$

One good property of a Kato potential V is that the heat semi-group associated to the Schrödinger operator $\Delta - V$ has a behaviour "close" to the one of the heat semi-group of the Laplacian, in the following sense (we refer to [Sim82, Theorem A.2.1] for the proof):

Proposition 3.1. *Consider $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq 0$. Then V is a Kato potential if and only if the operator*

$$e^{-t(\Delta - V)} : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n),$$

is bounded and satisfies

$$\lim_{t \to 0} \| e^{-t(\Delta - V)} - e^{-t\Delta} \|_{\infty,\infty} = 0. \tag{3.3}$$

This allows one to recover regularity properties for solutions of the Schrödinger equation $(\Delta - V)u = 0$, with a potential V which is not necessarily in L^p. Indeed, it is known that for $p > n/2$, functions in $L^p_{loc}(\mathbb{R}^n)$ are Kato potentials, see Example E in [Sim82], but the following gives an easy example of a Kato potential not belonging to $L^p(\mathbb{R}^n)$.

Example 3.2. It is proven in [Sim82, Proposition A.2.4] that a function $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq 0$ is a Kato potential if and only if the function

$$f(x) = \int_{B(x,1)} \frac{V(y)}{|x-y|^{n-2}}\,\mathrm{d}y,$$

is continuous. This in turns gives conditions for radial functions to be Kato potentials. In particular, the function

$$V(r) = r^{-2} \ln(r)^{-\alpha},$$

is a Kato potential for any $\alpha > 1$ and it does not belong to $L^p(\mathbb{R}^n)$ for any $p > \frac{n}{2}$.

3.3 Kato bounds on manifolds

Let (M^n, g) be a complete Riemannian manifold and Ric_g its Ricci curvature. We define the first eigenvalue of Ric_g at a point $x \in M$ by

$$\rho(x) = \inf_{v \in T_xM, g_x(v,v)=1} \mathrm{Ric}_{g_x}(v, v).$$

The negative part of Ric_g is then defined as the negative part of ρ:

$$\mathrm{Ric}_{\text{-}}(x) = \begin{cases} 0 & \text{if } \rho(x) \geq 0 \\ -\rho(x) & \text{otherwise.} \end{cases}$$

Observe that $\mathrm{Ric}_{\text{-}}$ is a non-negative function such that for any $x \in M$ we have $\mathrm{Ric}_{g_x} \geq -\mathrm{Ric}_{\text{-}}(x)$. We define the Kato constant of a manifold as the Kato constant of $\mathrm{Ric}_{\text{-}}$.

Definition 3.2 (Kato constant)**.** Let (M^n, g) a complete Riemannian manifold, Δ_g its Laplacian and $H : \mathbb{R}_+ \times M \times M \to \mathbb{R}_+$ its heat kernel. Let $t > 0$. The Kato constant of M at time t is given by

$$k_t(M, g) = \sup_{x \in M} \int_0^t \int_M H(s, x, y) \, \mathrm{Ric}_{\text{-}}(y) \, \mathrm{d}\,\mathrm{vol}_g(y) \, \mathrm{d}s.$$

In the case of closed manifolds, the analytic and geometric consequences of a bound on the Kato constant

$$k_T(M, g) \leq C_n, \tag{3.4}$$

for a positive dimensional constant C_n and a fixed time T have been studied in [Car19, Ros19, CR20, RS17, RW22]. Many similar properties to the ones of manifolds with non-negative Ricci curvature have been obtained: for instance, a lower bound on the first non-zero eigenvalue of the Laplacian [Car19], an upper bound on the first Betti number [Car19, Ros19], a Myers diameter theorem [CR20]. G. Carron proved pre-compactness in Gromov-Hausdorff topology of closed manifolds which satisfy (3.4) with $C_n = 1/16n$: this result represented the starting point to develop a regularity theory for limits of manifolds with a uniform bound on the Kato constant.

However, the scaling properties of the Kato constant make it clear that a bound as (3.4) may not be enough to obtain fine structure results. Indeed, the local geometry of a metric space can be studied in terms of *tangent cones*, which in the case of a limit of smooth manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha \in A}$ are limits of *rescaled* manifolds $\{(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha, x_\alpha)\}_{\alpha \in A}$, where ε_α goes to zero as α tends to infinity. In the case of a bound (3.4), the scaling properties of the Ricci tensor, heat kernel and volume measure lead to

$$k_t(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) = k_{\varepsilon_\alpha t}(M_\alpha, g_\alpha) < C_n. \tag{3.5}$$

This means that for sequences of rescaled manifolds, there is no improvement in assumption (3.4), in opposition to the case of a Ricci lower bound, where a tangent cone is a limit of manifolds with almost non-negative Ricci curvature.

We observe that in the Euclidean space, a function is a Kato potential if its Kato constant goes to zero as t tends to zero. As a consequence, when having a sequence of manifolds, the idea is to uniformly control not only their Kato constants at some fixed time T, but also the way in which they tend to zero with time. This motivated the introduction of the following conditions.

Definition 3.3 (Dynkin and Kato bounds)**.** Let $n \in \mathbb{N}$. A sequence of complete Riemannian manifolds $\{(M^n_\beta, g_\beta, o_\beta)\}_{\beta\in B}$ satisfies a Dynkin bound (D) if there exist $T > 0$ and $\gamma \in (0, 1/n-2)$ such that for all $\beta \in B$ we have

$$k_T(M_\beta, g_\beta) \le \gamma < \frac{1}{n-2}. \tag{D}$$

The sequence satisfies a Kato bound (K) if there exist $T > 0$ and a non-decreasing function $f : (0, T] \to \mathbb{R}_+$ such that

$$f(T) \le \frac{1}{3(n-2)} \tag{D'}$$

$$\lim_{t\to 0} f(t) = 0 \tag{3.6}$$

$$\text{for all } t \in (0, T], \quad \beta \in B, \quad k_t(M_\beta, g_\beta) \le f(t). \tag{3.7}$$

The Kato bound is said to be strong if moreover there exists $\Lambda > 0$ such that

$$\int_0^T \frac{f(t)}{t} dt \le \Lambda. \tag{SK}$$

In dimension 2, we say that a sequence of surfaces $\{M^2_\beta, g_\beta)\}_{\beta\in B}$ satisfies a Dynkin bound whenever there exists $T > 0$ such that for all $\beta \in B$ we have $k_T(M_\beta, g_\beta) < \infty$: we explain in Section 3.5 why in the case of surfaces we do not need any restriction on the bound on the Kato constant in order to obtain regularity results.

Conditions (3.6) and (3.7) together with the re-scaling property of the Kato constant ensure that for any $t \in (0, T]$ and $(\varepsilon_\beta)_\beta$ such that $\varepsilon_\beta \to 0$ we have

$$\lim_{\beta\to\infty} k_t(M_\beta, \varepsilon_\beta^{-2} g_\beta) \le \lim_{\beta\to\infty} f(\varepsilon_\beta t) = 0,$$

this giving hopes to obtain results on tangent cones.

As for (SK), it is usually called in the literature a "Dini condition", as it is similar to the Dini criterion for the convergence of Fourier series. When f is replaced by its modulus of continuity, this condition is used in different settings, from harmonic analysis and regularity theory to dynamical systems, see for instance [KZ22, KZ23, DPLM17, Bou24, LZ00, FJ01] and the references therein. We use this condition to define an appropriate almost monotone quantity based on the heat kernel and to show that, in the non-collapsed case, tangent cones are metric cones. Observe that the strong Kato bound is implied by the following assumptions.

1. A lower bound on the Ricci curvature. Indeed, the lower bound $\mathrm{Ric}_{g_\alpha} \geq -K$ implies $k_t(M_\alpha, g_\alpha) \leq Kt$, thus the strong Kato bound.

2. A smallness condition on the L^p norm of Ricci: a result of P. Stollmann and C. Rose [RS17] ensures that for any $p > \frac{n}{2}$ there exists $\varepsilon(n,p)$ such that if $\| \mathrm{Ric}_{\text{-}} \|_p < \varepsilon(n,p)$, then the strong Kato bound holds.

3. In dimension $n \geq 4$, if the scalar curvature is bounded and the Q-curvature is bounded from below (see [CMT24, Proposition 2.20]).

4. A Morrey bound on the negative part of Ricci: for any $x \in (M_\alpha, g_\alpha)$ and $r \in (0, \sqrt{T}/2]$
$$r^2 \fint_{B(x,r)} \mathrm{Ric}_{\text{-}} \, d\,\mathrm{vol}_{g_\alpha} \leq \Lambda \left(\frac{r}{\sqrt{T}} \right)^{\delta}.$$

3.4 Main results

In the following, we give an overview of the main results that we obtained in [CMT24, CMT22, CMT23b, CMT23a] for limits of manifolds.

In [CMT23a] we proved the following pre-compactness result for complete manifolds satisfying a Dynkin bound, extending the previous result of G. Carron.

Proposition 3.3. *Let $\{(M_\beta, g_\beta, o_\beta)\}_{\beta \in B}$ be a sequence of complete pointed Riemannian manifolds satisfying a Dynkin bound (D). Let $\mu_\beta = v_{g_\beta}(B(o_\beta, \sqrt{T}))^{-1} v_{g_\beta}$, where v_{g_β} is the Riemannian measure associated to g_β. Then there exist a subsequence $A \subset B$ and a pointed metric measure space (X, d, μ, o) such that $\{(M_\alpha, \mathsf{d}_{g_\alpha}, \mu_\alpha, o_\alpha)\}_{\alpha \in A}$ converges to (X, d, μ, o) in pointed measured Gromov-Hausdorff topology.*

The above pre-compactness put us in a position to define the following classes of metric measure spaces.

Definition 3.4. A metric measure space (X, d, μ, o) is called a Dynkin limit (respectively a Kato limit) if there exists a sequence of complete pointed Riemannian manifolds $\{(M_\alpha, g_\alpha, \mu_\alpha o_\alpha)\}_{\alpha \in A}$, where μ_α is defined as above, satisfying (D) for some $\gamma \in (0, 1/n-2), T > 0$ (respectively the Kato bound (K)) and converging to (X, d, μ, o).

Even in the case of a Dynkin bound, in [CMT23a] we were able to prove a structure result for Dynkin limits:

Theorem 3.4. *Let (X, d, μ, o) be a Dynkin limit. Then there exist $N > n$, $C, K > 0$ only depending on γ and n, a distance $\overline{\mathsf{d}}$ and a measure $\overline{\mu}$ such that $\mathsf{d} \leq \overline{\mathsf{d}} \leq C\mathsf{d}$, $\mu \leq \overline{\mu} \leq C\mu$ and the space $(X, \overline{\mathsf{d}}, \overline{\mu})$ is an $\mathrm{RCD}(-K/T, N)$ space.*

In other words, a Dynkin limit is bi-Lipschitz equivalent to an RCD space. In the case of surfaces, we obtained moreover that, without any restriction on the bound γ on the Kato constant, $(X, \overline{\mathsf{d}})$ is a 2-dimensional Alexandrov space with curvature bounded from below.

In [CMT24] we had previously shown that for a Kato limit of *closed* manifolds such that the Kato constant is bounded by $1/16n$ instead of $1/3(n-2)$, tangent cones are $\mathrm{RCD}(0,n)$ spaces, with the same n as the dimension of the manifolds. In [CMT22] we obtained a rectifiability result for these Kato limits. We improved both of these results in [CMT23a]. More precisely we have:

Corollary 3.5. *Let (X, d, μ, o) be a Dynkin limit. Then there exists $j \in \{1, \ldots, n\}$ such that X is j-rectifiable.*

The fact that a Dynkin limit carries a distance and measure that make it and $\mathrm{RCD}(K,N)$ space for some $N > n$, but it is j-rectifiable for some j not larger than n is due to the lower semi-continuity of the essential dimension proven in [Kit19, BPS21]. Observe that an $\mathrm{RCD}(K,N)$ space of essential dimension j is not always an $\mathrm{RCD}(K,j)$ space. Indeed, G. Wei and J. Pan [PW22] constructed an example of a collapsed Ricci limit which is $\mathrm{RCD}(0,N)$ for $N > 2$ and has essential dimension equal to 2. Observe that in this case the limit measure is not absolutely continuous with respect to the Hausdorff measure. We also refer to S. Honda's survey [Hon20a] and the conjectures therein for more on this subject.

The two previous results implies that tangent cones of a Dynkin limit are all $\mathrm{RCD}(0,N)$ spaces, not necessarily $\mathrm{RCD}(0,n)$, and moreover μ-almost everywhere unique, equal to the j-dimensional Euclidean space endowed with a multiple of the Hausdorff measure. However, in the case of Kato limits, we were able to obtain that all tangent cones at *any* point are $\mathrm{RCD}(0,n)$ spaces.

Corollary 3.6. *Let (X, d, μ, o) be a Kato limit. Then for all $x \in X$, any tangent cone $(Y, \mathsf{d}_Y, \mu_Y, x)$ of X at x is an $\mathrm{RCD}(0,n)$ space.*

Corollaries 3.5 and 3.6 extend the results obtained by J. Cheeger and T. Colding [CC00a, Theorem 5.7] and by T. Colding and A. Naber [CN12] in the case of Ricci limit spaces.

Another significant property that distinguishes Kato limits from Dynkin limits is that the pointed measured Gromov-Hausdorff convergence improves to the appropriate convergence of the Cheeger energies.

Theorem 3.7. *Let (X, d, μ, o) a Kato limit and $\{(M^n_\alpha, g_\alpha, \mu_\alpha, o_\alpha)\}$ be a sequence of manifolds satisfying a Kato bound, converging to (X, d, μ, o). For every $u \in C^1_c(M_\alpha)$ let*

$$\mathcal{E}_\alpha(u) = \int_M |du|^2 \, \mathrm{d}\mu_\alpha.$$

Then the Dirichlet energies $\{\mathcal{E}_\alpha\}_\alpha$ Mosco converge to the Cheeger energy Ch of (X, d, μ, o).

We refer to [CMT24, Section 1.4.3] for the precise definition of Mosco convergence. The previous statement was proven firstly in [CMT24], then extended to complete ones thanks to [CMT23a]. We point out that, for RCD spaces, pointed measured Gromov-Hausdorff convergence always implies Mosco convergence of the Cheeger energy, see

[GMS15, AH17]. In order to prove Theorem 3.7, we can exploit RCD theory even if a Kato limit is *not* an RCD space. We can also obtain energy convergence using an argument that depends on the Li-Yau inequality and on heat kernel convergence for Dirichlet spaces: this method does not need any synthetic curvature bound in the compact case. We refer to Section 3.6.3 for more details on both strategies of proof.

We obtained stronger structure results when assuming the strong Kato bound and non-collapsing. We define non-collapsed strong Kato limits as follows.

Definition 3.5. A pointed metric space (X, d, o) is a non-collapsed strong Kato limit if there exists a sequence of pointed complete Riemannian manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha \in A}$ that satisfies the strong Kato bound (SK), is non collapsed, that is, for some $v > 0$

$$v_{g_\alpha}(B(o_\alpha, \sqrt{T})) \geq v T^{\frac{n}{2}}, \tag{NC}$$

and converges in pointed Gromov-Hausdorff topology to (X, d, o).

Observe that in the definition of non-collapsed strong Kato limits we only consider pointed Gromov-Hausdorff convergence and we do not require convergence of the Riemannian measures: indeed, we proved that in this case the convergence automatically improves to *measured* Gromov-Hausdorff convergence. Together with this volume continuity, in [CMT24, CMT22, CMT23a] we proved the following regularity results.

Theorem 3.8. *Let (X, d, o) be a non-collapsed strong Kato limit. Then the following hold.*

1. *The Hausdorff dimension of X is equal to n.*

2. ***Tangent cones are metric cones.*** *For any $x \in X$ and for any tangent cone (Y, d_Y, x) at x there exists a metric space (Z, d_z) such that (Y, d_Y, x) is isometric to the metric cone over Z.*

3. ***Volume continuity.*** *For any $r > 0$ and $x_\alpha \in M_\alpha$ converging to x,*

$$v_{g_\alpha}(B(x_\alpha, r)) \to \mathcal{H}^n(B(x, r)).$$

4. ***Stratification.*** *For any $k \in \{0, \dots, n\}$ let*

$$\mathcal{S}^k(X) = \{x \in X \text{ s.t. for any tangent cone } (Y, d_Y, x) \nexists (Z, d_Z) \text{ s.t. } Y = \mathbb{R}^{k+1} \times Z\},$$

then $\dim_{\mathcal{H}}(\mathcal{S}^k(X)) \leq k$. Moreover $\mathcal{S}^{n-1}(X) = \emptyset$ and $\mathcal{H}^n(\mathcal{S}^{n-2}(X)) = 0$.

5. ***Reifenberg regularity.*** *Let $\mathcal{R} = X \setminus \mathcal{S}^{n-2}$. Then for any $\nu \in (0,1)$ there exists an open set $\mathcal{U}_\nu \subset X$ such that $\mathcal{R} \subset \mathcal{U}_\nu$ and $\mathcal{U}_\nu$ is bi-Hölder diffeomorphic to a C^ν-manifold.*

All the previous statements recover under a much weaker assumption what was known for Ricci limits thanks to Cheeger-Colding's theory [CC97, CC00a, CC00b] and for limits of manifolds with an L^p bound on the negative part of the Ricci curvature [Yan92a, Yan92b, PW97, PW01, DWZ18, Ket21]. Our proofs often rely on the appropriate comprehension of the behaviour of the heat kernel or the heat semi-group, and they take advantage of the recent tools for the convergence of Dirichlet spaces and the theory of RCD spaces. In some cases, we avoided arguments by contradiction, that are extensively used in Cheeger-Colding's theory and related work, see for instance [CN15, JN21, CJN21], and provided direct proofs, for example in showing that tangent cones are metric cones or when giving a quantitative version of Reifenberg regularity.

We conclude this review of our main results with a geometric application, obtained in [CMT23b], which generalizes J. Cheeger and T. H. Colding's torus stability result [CC00a, Col97] for manifolds with a small Kato constant.

Theorem 3.9 (Torus stability). *For any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon, n) > 0$ such that if (M^n, g) is a closed Riemannian manifold of diameter D satisfying*

$$b_1(M) = n \qquad \text{and} \qquad k_{D^2}(M^n, g) \leq \delta(n, \varepsilon),$$

then there exists an εD-Gromov-Hausdorff isometry between M and a flat torus. Moreover, let $f : [0,1] \to \mathbb{R}_+$ be a non-decreasing function satisfying (SK): then there exists $\delta(n, f)$ such that if

$$b_1(M) = n, \qquad k_{D^2}(M^n, g) \leq \delta(n, f),$$

and

$$k_{tD^2}(M^n, g) \leq f(t) \quad \text{for all } t \in (0,1],$$

then M is diffeomorphic to a flat torus.

We point out that since an L^p smallness condition on the Ricci curvature implies a strong Kato bound, the previous theorem applies for L^p bounds, for which such a topological result was previously unknown.

3.5 From Kato to RCD

In this section we present the main result of [CMT23a] and we briefly explain how this, combined with RCD theory, allows us to obtain results on Dynkin and Kato limits, and on smooth complete manifolds.

By using classical properties of Schrödinger operators, we showed that a bound on the Kato constant of a complete manifold implies the existence of a conformal change under which the obtained weighted manifold is an RCD space. More precisely, we have obtained:

Theorem 3.10. *Let (M^n, g) be a complete Riemannian manifold.*

1. *Assume that there exist $T > 0$ and $\gamma \in (0, 1/(n-2))$ such that*

$$k_T(M^n, g) \leq \gamma. \tag{D}$$

Then there exist $K \geq 0$, $C > 0$ and $N > n$ depending on n and γ only, and $f \in \mathcal{C}^2(M)$ with $0 \leq f \leq C$, such that the weighted Riemannian manifold $(M, e^{2f}g, e^{2f}v_g)$ is an $\mathrm{RCD}(-K/T, N)$ space.

2. *If moreover*

$$k_T(M^n, g) \leq \frac{1}{3(n-2)}, \tag{D'}$$

then the constants K, N and C can be chosen to be $K = 4k_T(M^n, g)$, $N = n + 4(n-2)^2 k_T(M^n, g)$ and $C = 4k_T(M^n, g)$.

The proof of the previous theorem relies on studying the properties of the Schrödinger operator $\Delta_g - \lambda \mathrm{Ric}_{\text{-}}$, for a carefully chosen parameter λ only depending on γ and n. We show that under the assumption (D), there exists $\beta > 0$ depending on γ and n and a function φ, bounded between 1 and $e^{\beta T}$, such that

$$\Delta_g \varphi - \lambda \mathrm{Ric}_{\text{-}} \varphi \geq -2\beta\varphi.$$

We then set $f = \lambda^{-1}\log(\varphi)$ and use the appropriate transformation rule to show that the weighted manifold $(M, e^{2f}g, e^{2f}v_g)$ satisfies a Bakry-Émery inequality.

In dimension 2, the existence of the function φ is guaranteed without any restriction on γ: whenever the Kato constant is finite, there exists a conformal metric with Gauss curvature bounded from below.

3.5.1 Consequences on Dynkin and Kato limits

Pre-compactness for manifolds satisfying a Dinkin bound, as stated in Proposition 3.3, directly follows from the previous statement. Indeed, the uniform bound on the conformal change given by Theorem 3.10 ensures that a complete manifold (M^n, g) which satisfies (D), seen as a metric measure space, is bi-Lipschitz equivalent to the corresponding weighted manifold $(M, e^{2f}g, e^{2f}v_g)$, which is an $\mathrm{RCD}(-K/T, N)$ space. Therefore, pre-compactness directly follows from the pre-compactness of RCD spaces with respect to pointed measured Gromov-Hausdorff convergence.

As for Theorem 3.4, a Dynkin limit (X, d, μ, o) is such there exists a sequence of pointed manifolds $\{(M_\beta, g_\beta, \mu_\beta, o_\beta)\}_{\beta \in B}$ satisfying (D) and converging to (X, d, μ, o). By Theorem 3.10, there exists a uniformly bounded sequence of positive functions $(f_\beta)_{\beta \in B}$ such that the weighted manifolds $(M_\beta, e^{2f_\beta}g_\beta, e^{2f_\beta}\mu_\beta, o_\beta)$ are $\mathrm{RCD}(-K/T, N)$ spaces, or, in the case of surfaces, have Gauss curvature bounded from below. We can then use pre-compactness of RCD spaces or of Alexandrov spaces in dimension 2, in order to obtain that the sequence of weighted manifolds converge to the RCD space $(X, \overline{\mathsf{d}}, \overline{\mu}, o)$, or to an Alexandrov space in dimension 2. The uniform boundedness of $(f_\beta)_{\beta \in B}$ implies the desired control of $\overline{\mathsf{d}}$ and $\overline{\mu}$.

As we recalled in the previous chapter, an $\mathrm{RCD}(K,N)$ space is k-rectifiable for some $k \in \{0,\dots,\lfloor N \rfloor\}$, the so-called essential dimension. But the essential dimension is lower semi-continuous with respect to pointed measured Gromov-Hausdorff convergence, thanks to [Kit19, BPS21]. Therefore, since a Dynkin limit is a limit of n-dimensional manifolds, its essential dimension must be not larger than n: this leads to Corollary 3.5.

In the case of Kato limits, observe that in the second point of Theorem 3.10 we have an explicit dependence of K and N on the Kato constant. This allows us to prove that all tangent cones at any point are $\mathrm{RCD}(0,n)$ spaces. As this property was not explicitly proven in [CMT23a], we give a proof here.

Proof of Corollary 3.6. By definition of a Kato limit and of tangent cones, there exist a sequence of manifolds $\{(M^n_\alpha, g_\alpha, \mu_\alpha, o_\alpha\}_{\alpha\in A}$ satisfying the Kato bound of Definition 3.3 and converging to (X, d, μ, o), a sequence of points $\{x_\alpha\}_{\alpha\in A}$ and $\{\varepsilon_\alpha\}_{\alpha\in A}$, $\varepsilon_\alpha > 0$ and $\varepsilon_\alpha \to 0$ such that the re-scaled sequence $\{(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha, \mu_\alpha(B(x_\alpha,\varepsilon_\alpha))^{-1}\mu_\alpha, x_\alpha\}_{\alpha\in A}$ converges to $(Y, \mathsf{d}_Y, \mu_Y, x)$. Denote by $\widetilde{d}_\alpha$ and $\widetilde{\mu}_\alpha$ the distance and measure of each rescaled manifold. We know that

$$k_T(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) = k_{\varepsilon_\alpha^2 T}(M_\alpha, g_\alpha) \le f(\varepsilon_\alpha^2 T) \le \frac{1}{3(n-2)}. \tag{3.8}$$

Then we can apply Theorem 3.10 to each rescaled manifold: we find $u_\alpha \in \mathcal{C}^2(M_\alpha)$ such that

$$0 \le u_\alpha \le 4k_T(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha), \tag{3.9}$$

and $(M_\alpha, e^{2u_\alpha}\varepsilon_\alpha^{-2} g_\alpha, e^{2u_\alpha}\mu_\alpha(B(x_\alpha,\varepsilon_\alpha))^{-1}\mu_\alpha)$ is an $\mathrm{RCD}(-K_\alpha/T, N_\alpha)$ space, where

$$K_\alpha = 4k_T(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) \quad N_\alpha = n + 4(n-2)^2 k_T(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha). \tag{3.10}$$

Denote by $\bar{\mathsf{d}}_\alpha$ the distance associated to $e^{2u_\alpha}\varepsilon_\alpha^{-2} g_\alpha$ and by $\overline{\mu}_\alpha = e^{2u_\alpha}\mu_\alpha(B(x_\alpha,\varepsilon_\alpha))^{-1}\mu_\alpha$. Thanks to the uniform Kato bound, we know that $(M_\alpha, \bar{\mathsf{d}}_\alpha, \overline{\mu}_\alpha)$ is an $\mathrm{RCD}(-4f(\varepsilon_\alpha^2 T)/T, n + 4(n-2)^2 f(\varepsilon_\alpha^2 T))$ space. Since f tends to zero in zero, for any $\nu > 0$ there exists α_ν large enough such that for all $\alpha \ge \alpha_\nu$ the weighted manifolds $(M_\alpha, \bar{\mathsf{d}}_\alpha, \overline{\mu}_\alpha)$ are $\mathrm{RCD}(-4\nu/T, n + 4(n-2)^2\nu)$ spaces. By pre-compactness of RCD spaces and using Theorem 3.4, for any $\nu > 0$ the tangent cone Y is endowed with a distance d_ν and a measure μ_ν such that

$$\mathsf{d}_Y \le \mathsf{d}_\nu \le e^{4\nu}\mathsf{d}_Y \quad \text{and} \quad \mu_Y \le \mu_\nu \le e^{4\nu}\mu_Y,$$

$(Y, \mathsf{d}_\nu, \mu_\nu, x)$ is an $\mathrm{RCD}(-4\nu/T, n + 4(n-2)^2\nu)$ space and the weighted manifolds $(M_\alpha, \bar{\mathsf{d}}_\alpha, \overline{\mu}_\alpha, x_\alpha)$ converge to $(Y, \mathsf{d}_\nu, \mu_\nu, x)$. But as ν tends to zero, d_ν and μ_ν converge to d_Y and μ_Y, so that $(Y, \mathsf{d}_Y, \mu_Y, x)$ is an $\mathrm{RCD}(0,n)$ space. □

3.5.2 Consequences on manifolds

Theorem 3.10 allows us to obtain the following results on a complete manifold (M^n, g) satisfying (D):

- the Riemannian measure is doubling: there exists θ only depending on n and γ such that for all $x \in M$ and $r \in (0, \sqrt{T}/2)$, we have $v_g(B(x, 2r)) \leq \theta v_g(B(x, 2r))$;
- an L^1 Poincaré inequality holds with a constant λ only depending on n and γ;
- there exist good cut-off functions, that is, for any $x \in M$ and $r \in (0, \sqrt{T})$ there exists $\chi_{x,r}$ equal to one on $B(x, r/2)$, vanishing outside $B(x, r)$ and such that for some constant $C(n, \gamma)$ we have

$$|d\chi_{x,r}|^2 + |\Delta_g \chi_{x,r}| \leq \frac{C(n,\gamma)}{r^2}.$$

We refer to Section 4 in [CMT23a] for the precise statements of the above properties.

We also proved the following almost monotonicity for the volume ratio.

Theorem 3.11. *Let $T > 0$ and $f : (0, T] \to \mathbb{R}_+$ (M^n, g) be a non-decreasing function satisfying (3.6) and (SK). Let (M^n, g) be a complete manifold such that for all $t \in (0, T]$ we have $k_t(M^n, g) \leq f(t)$. Then for any $x \in M$, $R \in (0, \sqrt{T}]$, $\eta \in (1/\sqrt{2}, 1)$ and $r \in (0, \eta R)$ we have*

$$\frac{v_g(B(x,R))}{R^n} \leq \exp\left(-\frac{C(n)}{\log(\eta)} \int_r^R \frac{f(s^2)}{s}\, \mathrm{d}s\right) \frac{v_g(B(x,r))}{r^n}.$$

This allows us to show that on a non-collapsed strong Kato limit (X, d, o), the volume density

$$\rho_X(x) = \lim_{r \to 0^+} \frac{\mathcal{H}^n(B(x,r))}{\omega_n r^n},$$

is well-defined at any point $x \in X$. We refer to Section 3.6.4 for a discussion about monotone quantities and their role in proving regularity for non-collapsed strong Kato limits.

We conclude this section by observing that in the compact case and under the slightly stricter bound

$$k_T(M^n, g) \leq \frac{1}{16n}, \tag{Dyn}$$

Theorem 3.10 is not needed to obtain neither pre-compactness, nor the doubling property of the measure [CMT24, Proposition 2.3] (see also Propositions 3.8 and 3.11 in [Car19]). For compact manifolds (M^n, g) satisfying (Dyn), we used a Li-Yau inequality originally obtained in [Car19] to obtain the doubling property of the Riemannian measure v_g: pre-compactness then follows directly from the well-known result of Gromov, without relying on RCD theory.

3.6 Strategies of chosen proofs

In this section we present the main tools and techniques that we used in the proofs of [CMT24, CMT22, CMT23b]: in those articles, we considered closed manifolds. Thanks to the existence of good cut-off functions for complete manifolds satisfying (D), our strategies can be extended from only closed to complete manifolds. One of the interests of our approach is that it is based on convergence results for Dirichlet spaces and in particular on heat kernel convergence, without assuming a Ricci lower bound. As pointed out by N. Gigli in [Gig23, Section 4], this can also be used to obtain stability of the heat flow, for which the known proof in the setting of CD spaces relies on optimal transport.

A key point in many of our proofs is a Li-Yau inequality, that holds under the assumption that the Kato constant is smaller than $1/16n$, where n is the dimension of the manifold. As a consequence, throughout this section we assume (M^n, g) to be a complete manifold that for some $T > 0$ satisfies (Dyn).

3.6.1 The Li-Yau inequality

In [Car19, Proposition 3.3], Carron showed a Li-Yau inequality for positive solutions of the heat equation on a compact manifold such that (Dyn) holds. As we pointed out in [CMT23a, Remark 4.5], this inequality can be extended to complete manifolds and improved to obtain the following.

Theorem 3.12. *Let (M^n, g) be a complete manifold such that for some $T > 0$ we have (Dyn). Let $u : [0, T] \times M \to \mathbb{R}_+$ be a positive solution of the heat equation $(\partial_t + \Delta)u = 0$. Then on $[0, T] \times M$ we have*

$$e^{-5k_t(M,g)}|\nabla \log u|^2 - \partial_t \log u \leq \frac{n}{2t} e^{5k_t(M,g)}. \tag{LY}$$

For closed manifolds, we obtained (LY) with the same proof as in [Car19, Proposition 3.3], by choosing the parameter δ to be equal to $k_T(M, g)^2 n^2/(n + 1)$. The complete case is obtained thanks to the existence of good cut-off functions proven in [CMT23a]. We point out that in the first version of this inequality in [Car19], the exponent in the exponentials contained the square root of $k_t(M^n, g)$ instead of just $k_t(M^n, g)$.

Remark 3.13. The reason for having $16n$ in the denominator is technical and depends on the proof of Lemma 3.2 in [Car19].

The previous inequality gives a first intuition, independent of RCD spaces, of the reason why tangent cones of Kato limits should have non-negative Ricci curvature in a generalized sense. Indeed, a tangent cone at a point x of a Kato limit (X, d, μ, o) is a metric measure space $(Y, \mathsf{d}_Y, \mu_Y, x)$ obtained as a limit of rescaled manifolds $(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha, \mathrm{vol}_{g_\alpha}(B(x_\alpha, \varepsilon_\alpha))^{-1} \mathrm{vol}_{g_\alpha}, x_\alpha)$, where ε_α tends to 0 and x_α tends to x as α goes to infinity. For these rescaled manifolds, for any $t \in [0, T]$ we have

$$k_t(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) \to 0 \text{ as } \alpha \to \infty,$$

and moreover the Li-Yau inequality (LY) holds. As a consequence, if we were able to pass to the limit in inequality (LY), the exponential terms would tend to 1, and we would obtain the following inequality on tangent cones

$$|\nabla \log u|^2 - \partial_t \log u \leq \frac{n}{2t}. \tag{LY0}$$

This latter inequality coincides with the classic Li-Yau inequality on manifolds of dimension n with non-negative Ricci curvature; it also holds on $\mathrm{RCD}^*(0,n)$ spaces thanks to [GM14]. Thus tangent cones of Kato limits should have non-negative Ricci curvature in some sense. In order to make this intuition rigorous we need the appropriate notion of heat kernel convergence. The setting in which we developed our theory is the one of PI-Dirichlet spaces, that is, Dirichlet spaces whose measure is doubling and that carry a Poincaré inequality. As we recalled in the previous section, manifolds satisfying (D) have these properties. For sequences of (uniformly) PI-Dirichlet space, we have good notions of convergence for the energies and heat kernel: we present below the pre-compactness theorem that we need.

3.6.2 PI-Dirichlet spaces, energy and heat kernel convergence

In the following, we give some essential background on PI-Dirichlet spaces and heat kernel convergence: we refer to [CMT24, Section 1] for the details.

Definition 3.6. Fix $\theta, \gamma, R > 0$. A $\mathrm{PI}_{\theta,\gamma}(R)$ Dirichlet space is a regular, strongly local Dirichlet space $(X, \mathsf{d}, \mu, \mathcal{E})$ such that

- (X, d, μ) is θ-doubling at scale R, that is, for all $x \in X$ and $r \in (0, R/2]$
$$\mu(B(x, 2r) \leq \theta\mu(B(x, r));$$
- for any $r \in (0, R]$ and ball B of radius r, for any $u \in \mathcal{D}(\mathcal{E})$ the following Poincaré inequality holds
$$\left\| u - \fint_B u \right\|_{L^2(B,\mu)} \leq \gamma r^2 \int_B \mathrm{d}\Gamma(u, u), \tag{3.11}$$
where Γ is the carré du champ associated to $\mathcal{E}$.

As we recalled in the previous chapter, any Dirichlet space $(X, \mathsf{d}, \mu, \mathcal{E})$ carries a non-negative definite self-adjoint operator L associated to the Dirichlet form $\mathcal{E}$. Moreover, L generates an analytic sub-Markoviam semigroup $(P_t := e^{-tL})_{t>0}$ acting on $L^2(X, \mu)$ and satisfying
$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} P_t f = -L(P_t f) & \forall t > 0, \\ \lim\limits_{t \to 0} \|P_t f - f\|_{L^2(X,\mu)} = 0. \end{cases}$$
We usually refer to $\{P_t\}_{t>0}$ as the heat semi-group. An important property of a PI-Dirichlet space is that it also carries a Hölder continuous *heat kernel*, that is, $H : (0, \infty) \times M \times M \to \mathbb{R}$ satisfying for any $f \in L^2(X, \mu)$, $t > 0$ and μ-a.e. $x \in X$
$$P_t f(x) = \int_X H(t, x, y) f(y) \, \mathrm{d}\mu(y).$$

In order to state the pre-compactness result that we need, we also introduce the *intrinsic distance* associated to the carré du champ.

Definition 3.7. Let $(X, d, \mu, \mathcal{E})$ be a regular, strongly local Dirichlet space. The *intrinsic* extended pseudo-distance $\mathsf{d}_{\mathcal{E}}$ associated with $\mathcal{E}$ is defined by

$$\mathsf{d}_{\mathcal{E}}(x,y) := \sup\{|f(x)-f(y)| \,:\, f \in \mathcal{C}(X) \cap \mathcal{D}_{loc}(\mathcal{E}) \text{ s.t. } \Gamma(f) = \rho\mu,\ |\rho| \le 1\ \mu\text{-a.e. on } X\}.$$

The pre-compactness result for PI-Dirichlet spaces then reads as follows (we refer to [Kas05, KS03] for earlier versions of this statement).

Theorem 3.14. *Let $\eta > 1$, $\theta, \gamma, R > 0$ and $\{(X_\beta, \mathsf{d}_\beta, \mu_\beta, \mathcal{E}_\beta, o_\beta)\}$ be $\mathrm{PI}_{\theta,\gamma}(R)$ Dirichlet spaces such that*

$$\eta^{-1} \le \mu_\beta(B(o_\beta, R)) \le \eta.$$

Then there exist a subsequence $A \subset B$ and a regular, strongly local Dirichlet space $(X, \mathsf{d}, \mu, \mathcal{E}, o)$ such that $\{(X_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha)\}_{\alpha \in A}$ converges to (X, d, μ, o). Moreover the following holds.

1. *The Dirichlet space $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E}, o)$ is a $\mathrm{PI}_{\theta,\gamma}(R)$ space.*
2. *The Dirichlet forms $(\mathcal{E}_\alpha)_\alpha$ Mosco converge to $\mathcal{E}$. In particular, for any $t > 0$ the heat kernels $H_\alpha(t,\cdot,\cdot)$ converge uniformly on compact sets to the heat kernel $H(t,\cdot,\cdot)$ of $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E}, o)$.*
3. *There exists $c > 0$ such that $c\mathsf{d}_{\mathcal{E}} \le \mathsf{d} \le \mathsf{d}_{\mathcal{E}}$.*

The parameters θ and γ must be the same for *all* spaces in the sequence. We can apply this result to sequences of manifolds $\{(M^n_\beta, g_\beta, v_{g_\beta}, \mathcal{E}_\beta, o_\beta)\}_{\beta \in B}$ which satisfy (D) because the constants appearing in the doubling property and the Poincaré inequality only depend on the dimension n of the manifolds (see [CMT23a, Section 4] and [CMT24, Proposition 2.3]).

We underline that there is a priori a substantial difference between the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ associated to the limit energy $\mathcal{E}$ and the limit distance d obtained through the Gromov-Hausdorff convergence: even if they are bi-Lipschitz equivalent, the limit space is a PI-Dirichlet space only when endowed with the intrinsic distance. The heat kernel is then associated to $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ and *not* to $(X, \mathsf{d}, \mu, \mathcal{E})$. Moreover, there are explicit examples for which the intrinsic distance and the limit distance do not coincide. In [ACT21], the authors constructed a sequence of conformal metrics on the flat tori whose associate distances converge in Gromov-Hausdorff topology to a Finsler metric. More precisely, consider the standard flat torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$, where $\Gamma = (2\pi\mathbb{Z})^n$, with the Euclidean metric g_0. For any integer $\ell > 1$ define the function f_ℓ by

$$e^{nf_\ell}(x_1, \dots, x_n) = 1 - \frac{1}{2}\cos(\ell x_1),$$

and the conformal metrics $g_\ell = e^{2f_\ell} g_0$. Set $\mu_\ell = \mathrm{vol}_{g_\ell}$ and $\mathcal{E}_\ell$ as in (2.2). It is possible to show that the sequence $\{(\mathbb{T}^n, \mathsf{d}_{g_\ell}, \mu_\ell, \mathcal{E}_\ell)\}_\ell$ satisfies the assumptions of the previous theorem, therefore it converges up to a subsequence to a pointed metric measure space (X, d, μ, o), and $\mathcal{E}_\ell$ Mosco converges to a Dirichlet form $\mathcal{E}$. But, as proven in [ACT21, Theorem 8.1], the limit distance d is a Finsler metric: if d coincided with $\mathsf{d}_{\mathcal{E}}$, then d would be associated to a quadratic form, which is impossible for a Finsler metric unless it is a Riemannian metric.

3.6.3 Energy convergence for Kato limits

In this section we illustrate two ways of proving the convergence of energy stated in Theorem 3.7, that is, of showing that the Dirichlet energy in the limit coincides with the Cheeger energy associated to the limit distance. The first method is an application of the Li-Yau inequality (LY) and of heat kernel convergence; the second relies on Theorem 3.10 and RCD theory.

Energy convergence via Li-Yau inequality

Our first proof of energy convergence is contained in [CMT24, Section 4]: we briefly sketch its main arguments and refer to [CMT24, Section 4] for the details. The first step consists in showing that for a Kato limit $(X, \mathsf{d}, \mu, \mathcal{E}, o)$, where the Dirichlet form $\mathcal{E}$ is well defined thanks to Theorem 3.14, the distances d and $\mathsf{d}_{\mathcal{E}}$ are the same.

Proposition 3.15. *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato limit with $f(T) \leq \frac{1}{16n}$. Then $\mathsf{d} = \mathsf{d}_{\mathcal{E}}$.*

Sketch of the proof. Let $\{(M_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_{\alpha \in A}$ be a sequence of smooth manifolds satisfying the Kato bound and converging to $(X, \mathsf{d}, \mu, o, \mathcal{E})$. We denote by H_α and H the heat kernels of M_α and of $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ respectively. We fix $t > 0$, $x, y \in X$ and $\{x_\alpha, y_\alpha\}_{\alpha \in A}$ sequences of points in M_α converging respectively to $x, y \in X$. The idea of the proof is to use the Li-Yau inequality (LY) to show that for any $t \in (0, T)$, $\eta \in (0, 1)$

$$\log\left(\frac{H_\alpha(\eta t, x_\alpha, y_\alpha)}{H_\alpha(t, x_\alpha, y_\alpha)}\right) \leq \frac{ne}{2} \log\left(\frac{1}{\eta}\right) + \frac{\mathsf{d}_\alpha^2(x_\alpha, y_\alpha)}{4(1-\eta)t} e^{5f(t)}. \tag{3.12}$$

Thanks to heat kernel convergence and the convergence of the distances d_α to d, we obtain the analogue inequality on X: for all $x, y \in X$, $t \in (0, T)$, $\eta \in (0, 1)$

$$\log\left(\frac{H(\eta t, x, y)}{H(t, x, y)}\right) \leq \frac{ne}{2} \log\left(\frac{1}{\eta}\right) + \frac{\mathsf{d}^2(x, y)}{4(1-\eta)t} e^{5f(t)}. \tag{3.13}$$

We use Varadhan's formula

$$\mathsf{d}_{\mathcal{E}}^2(x, y) = \lim_{t \to 0}(-4t \log H(t, x, y)), \tag{3.14}$$

multiply (3.13) by $-4t$ and pass to the limit as t goes to 0 to obtain for any $\eta \in (0, 1)$

$$\mathsf{d}_{\mathcal{E}}^2(x, y) \leq \frac{\mathsf{d}(x, y)^2}{1-\eta}.$$

Passing to the limit as η goes to zero we get $\mathsf{d}_{\mathcal{E}} \leq \mathsf{d}$. By Theorem 3.14 we know $\mathsf{d}_{\mathcal{E}} \geq \mathsf{d}$, then the desired equality follows.

We explain how (3.12) is obtained from the Li-Yau inequality. Let u be a positive solution of the heat equation on M_α for α fixed, $t \in (0, T]$, $s \in (0, t)$ and $\gamma : [0, t-s] \to M_\alpha$ a minimizing geodesic from y_α to x_α. Define for any $\tau \in [0, t-s]$

$$\phi(\tau) = \log u(t - \tau, \gamma(\tau)),$$

and differentiate in τ:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\phi(\tau) &= -\frac{1}{u}\frac{\partial u}{\partial t} + \langle \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau), d\log u\rangle \\ &\leq \frac{ne^{5k_{t-\tau}}}{2(t-\tau)} - e^{-5k_t}\frac{|du|^2}{u^2} + \langle \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau), d\log u\rangle \\ &= \frac{ne^{5k_{t-\tau}}}{2(t-\tau)} - \left| e^{-\frac{5}{2}k_t} d\log u - \frac{e^{\frac{5}{2}k_t}}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau)\right|^2 + \frac{e^{5k_t}}{4}\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau)\right|^2 \\ &\leq \frac{ne^{5k_{t-\tau}}}{2(t-\tau)} + \frac{e^{5k_t}\mathsf{d}_\alpha^2(x_\alpha,y_\alpha)}{4(t-s)^2},\end{aligned}$$

wherein the first inequality we used (LY). We then integrate between 0 and $t-s$ and obtain:

$$\log\left(\frac{u(s,x)}{u(t,y)}\right) \leq \frac{n}{2}e\log\left(\frac{t}{s}\right) + \frac{e^{5f(t)}\mathsf{d}_\alpha^2(x_\alpha,y_\alpha)}{4(t-s)},$$

where we used $k_t(M_\alpha, g_\alpha) \leq f(t) \leq \frac{1}{16n}$ for any $t \in (0,T]$ and $\alpha \in A$. Choosing $s = \eta t$ and $u(\tau, z) = H_\alpha(\tau, x_\alpha, z)$ in the previous inequality directly leads to (3.12). □

Theorem 3.16. *Let $(X, \mathsf{d}, \mu, o, \mathcal{E})$ be a Kato limit. Then $\mathsf{Ch}_{\mathsf{d}} = \mathcal{E}$.*

Sketch of the proof. We use the following result of Koskela, Shanmugalingam and Zhou [KSZ14, Theorem 4.1]: let $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ be a PI Dirichlet space and assume that there exists a locally bounded function $h : [0,T] \to [0,+\infty)$ such that

$$\liminf_{t\to 0} h(t) = 1,$$

and for all $u \in \mathcal{D}(\mathcal{E})$, $\varphi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_c(X)$, $\varphi \geq 0$ we have

$$\int_X \varphi \,\mathrm{d}\Gamma(P_t u) \leq h(t)\int_X P_t\varphi \,\mathrm{d}\Gamma(u). \tag{3.15}$$

Then $\mathsf{Ch}_{\mathsf{d}_{\mathcal{E}}} = \mathcal{E}$. By the previous proposition we know that $\mathsf{Ch}_{\mathsf{d}} = \mathsf{Ch}_{\mathsf{d}_{\mathcal{E}}}$: our goal is therefore to show the existence of a function h such that (3.15) holds.

Denote by $\{(M_\alpha, \mathsf{d}_\alpha, \mu_\alpha, o_\alpha, \mathcal{E}_\alpha)\}_{\alpha\in A}$ a sequence converging to $(X, \mathsf{d}, \mu, o, \mathcal{E})$. Let $(P_t^\alpha)_{t>0}$ the heat semi-group of M_α and $\{u_\alpha\}_\alpha, \{\varphi_\alpha\}_\alpha$ two sequences such that $u_\alpha \in \mathcal{D}(\mathcal{E}_\alpha)$, $\varphi_\alpha \geq 0$ and $\varphi_\alpha \in \mathcal{D}(\mathcal{E}_\alpha) \cap \mathcal{C}_c(M_\alpha)$, u_α converges in energy to u and φ_α converges uniformly to φ (see Section 1.4.2 in [CMT24] for the precise definitions of convergence for functions). By choosing the appropriate gauging function and using Bochner inequality, we showed that

$$\int_{M_\alpha} \varphi_\alpha |dP_t^\alpha u_\alpha|^2 \,\mathrm{d}v_{g_\alpha} \leq e^{4f(t)}\int_{M_\alpha} P_t^\alpha\varphi_\alpha |du_\alpha|^2 \,\mathrm{d}v_{g_\alpha},$$

see Lemmas 4.3 and 4.4 and Corollary 4.5 in [CMT24]. By passing to the limit in the previous inequality we are able to obtain (3.15) with $h(t) = e^{4f(t)}$, thus to conclude. □

Energy convergence via Theorem 3.10

Energy convergence for Kato limits can also be deduced using the second point in Theorem 3.10. Consider a Kato limit (X, d, μ, o) of manifolds $\{(M_\alpha, g_\alpha, \mu_\alpha, o_\alpha)\}_{\alpha\in A}$, endowed with the Dirichlet energy $\mathcal{E}$ obtained as limit of the energies $\mathcal{E}_\alpha$.

By Theorem 3.10, for any $t \in (0, T]$ and $\alpha \in A$ there exists a function $u_{\alpha,t}$ such that

$$0 \le u_{\alpha,t} \le 4k_t(M_\alpha, g_\alpha),$$

and for $g_{\alpha,t} = e^{2u_{\alpha,t}} g_\alpha$, $\mu_{\alpha,t} = e^{2u_{\alpha,t}}\mu_\alpha$ the weighted manifold $(M_\alpha, \mathsf{d}_{g_{\alpha,t}}, \mu_{\alpha,t})$ is an $\mathrm{RCD}(-4k_t(M_\alpha, g_\alpha)/t, n + 4(n-2)^2 k_t(M_\alpha, g_\alpha))$ space. Thanks to the Kato bound, each weighted manifold is actually an $\mathrm{RCD}(-4f(t)/t, n+4(n-2)^2 f(t))$ space. Therefore, we can apply pre-compactness of RCD spaces to deduce that X is endowed with a distance d_t and measure μ_t such that $(M_\alpha, \mathsf{d}_{g_{\alpha,t}}, \mu_{\alpha,t}, o_\alpha)$ converges to $(X, \mathsf{d}_t, \mu_t, o)$, and (X, d_t, μ_t) is an $\mathrm{RCD}(-4f(t)/t, n + 4(n-2)^2 f(t))$ space. Moreover, by energy convergence for RCD spaces, the Dirichlet energies $\mathcal{E}_{\alpha,t}$ associated to $g_{\alpha,t}$ and $\mu_{\alpha,t}$ Mosco converge to the Cheeger energy $\mathsf{Ch}_{\mathsf{d}_t}$, whose intrinsic distance coincides eventually with d_t. It is not difficult to see that our conformal change does not change the Dirichlet energies: we have $\mathcal{E}_\alpha = \mathcal{E}_{\alpha,t}$ and thus $\mathcal{E} = \mathsf{Ch}_{\mathsf{d}_t}$. Thanks to Theorem 3.4 we also know that

$$\mathsf{d} \le \mathsf{d}_t \le e^{4f(t)}\mathsf{d} \quad \text{and} \quad \mu \le \mu_t \le e^{4f(t)}\mu.$$

Since $f(t)$ tends to zero as t goes to zero, from the inequality on the distances we deduce that d_t converges to d as t goes to zero. From the inequality on the measures, the definition of the intrinsic distance and the fact that $\mathcal{E} = \mathsf{Ch}_{\mathsf{d}_t}$, for t small enough we obtain that

$$\mathsf{d}_{\mathcal{E}} \le \mathsf{d}_t \le (1 + \varepsilon(t))\mathsf{d}_{\mathcal{E}},$$

where $\varepsilon(t)$ tends to zero as t goes to zero. As a consequence, d_t tends to the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ associated to $\mathcal{E}$. Therefore, the limit distance d coincides with the intrinsic distance $\mathsf{d}_{\mathcal{E}}$, and the limit Dirichlet energy $\mathcal{E}$ is equal to the Cheeger energy Ch_{d}.

3.6.4 Strong Kato bound, monotone quantities and regularity

In this section we focus on complete manifolds that satisfy a strong Kato bound, and we explain the main ideas and tools that lead to prove two structure results for non-collapsed strong Kato limits: tangent cones are metric cones and Reifenberg regularity. Thanks to the almost monotonicity result for the volume ratio that we proved in Theorem 3.11, we know that on a non-collapsed strong Kato limit the volume density is well-defined. This could allow us to use similar arguments to the ones of Cheeger-Colding theory in order to obtain regularity results. Here we prefer to underline the novelty and differences of our approach with respect to the previous literature. The main idea consists in introducing new monotone quantities based on the heat kernel, which are globally defined, instead of relying on the volume ratio.

We refer to [CMT24] and [CMT22]: even if in those papers we restricted ourselves to closed manifolds, thanks to the existence of good cut-off functions under a Dynkin bound, all of the following results hold and are stated here for complete manifolds.

Monotone quantities

Monotone quantities and non-negative Ricci curvature. When dealing with manifolds (M^n, g) such that $\mathrm{Ric}_g \geq 0$, there are several well-known monotone quantities.

- The *volume ratio*, defined for any $x \in M$ and $r > 0$ by

$$\mathcal{V}(x,r) = \frac{\mathrm{vol}_g(B(x,r))}{\omega_n r^n},$$

where ω_n is the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. The Bishop-Gromov inequality ensures that the map $r \to \mathcal{V}(x,r)$ is monotone non-increasing for all $x \in M$.

- The *on-diagonal heat kernel ratio*, defined for any $x \in M$ and $t > 0$ by

$$\mathcal{H}(x,t) = (4\pi t)^{\frac{n}{2}} H(t,x,x).$$

Observe that $(4\pi t)^{\frac{n}{2}}$ is the inverse of the on-diagonal heat kernel on $\mathbb{R}^n$. Thanks to the Li-Yau inequality (LY0), it is easy to show that the map $t \mapsto \mathcal{H}(t,x)$ is non-decreasing for any $x \in M$.

- The *Huisken entropy*, defined for any $x \in M$ and $s > 0$ by

$$\Theta_x(s) = \frac{1}{(4\pi s)^{\frac{n}{2}}} \int_M \exp\left(-\frac{\mathsf{d}_g(x,y)^2}{4s}\right) dv_g(y),$$

which has been shown to be monotone non-increasing in s by W. Jiang and A. Naber [JN21].

In [CMT24] we introduced a new family of monotone quantities depending on the heat kernel. Let (M^n, g) be a complete manifold with $\mathrm{Ric}_g \geq 0$ and introduce the function $U : (0,+\infty) \times M \times M \to (0,+\infty)$ such that for any $t > 0$ and $x, y \in M$

$$H(t,x,y) = \frac{\exp\left(-\frac{U(t,x,y)}{4t}\right)}{(4\pi t)^{\frac{n}{2}}}.$$

Now fix $x \in M$, $s,t > 0$ and define

$$\theta_x(s,t) = \int_M \frac{\exp\left(-\frac{U(t,x,y)}{4s}\right)}{(4\pi s)^{\frac{n}{2}}} dv_g(y).$$

Observe that by Varadhan's formula

$$\lim_{t\to 0} U(t,x,y) = \mathsf{d}(x,y)^2,$$

thus for any $x \in M$, when t tends to 0 we obtain $\theta_x(s,0) = \Theta(x,s)$. Consider the map $\lambda \mapsto \theta_x(\lambda s, \lambda t)$. Whenever $s = t$ it is constant equal to one, because of the stochastic completeness of the manifold:

$$\theta_x(\lambda t, \lambda t) = \int_M H(\lambda t, x, y)\, \mathrm{d}v_g(y) = 1.$$

Let $t > 0$. Then we have

$$\begin{aligned} (4\pi t)^{\frac{n}{2}} H(t,x,x) &= (4\pi t)^{\frac{n}{2}} \int_M H\left(\frac{t}{2}, x, y\right)^2 \mathrm{d}v_g(y) \\ &= \int_M (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{U(t/2,x,y)}{t}\right) \mathrm{d}v_g \\ &= \theta_x\left(\frac{t}{4}, \frac{t}{2}\right), \end{aligned}$$

where we used the semi-group law in the first equality and the definition of U in the second. As a consequence, for these choices of t and s we obtain

$$\lambda \mapsto \theta_x\left(\lambda\frac{t}{4}, \lambda\frac{t}{2}\right) = (4\pi\lambda t)^{\frac{n}{2}} H(\lambda t, x, x), \tag{3.16}$$

and this map in non-decreasing in λ.

Let $s > 0$ and $t = 0$. Then by Cavalieri's principle and the appropriate change of variables we have

$$\begin{aligned} \theta_x(s,0) &= \frac{1}{(4\pi s)^{\frac{n}{2}}} \int_M e^{-\frac{\mathrm{d}(x,y)^2}{4s}}\, \mathrm{d}v_g(y) \\ &= \frac{1}{(4\pi s)^{\frac{n}{2}}} \int_0^{+\infty} \mathrm{vol}_g(\{e^{-\frac{d^2(x,\cdot)}{4s}} > \tau\})\, \mathrm{d}\tau \\ &= \frac{1}{(4\pi s)^{\frac{n}{2}}} \int_0^{1} \mathrm{vol}_g(B(x, \sqrt{-4s\log(\tau)})\, \mathrm{d}\tau \\ &= \frac{1}{2}\frac{1}{(4\pi s)^{\frac{n}{2}}} \int_0^{+\infty} \mathrm{vol}_g(B(x,\rho\sqrt{s}))e^{-\frac{\rho^2}{4}}\rho\, \mathrm{d}\rho \\ &= \frac{\omega_n}{2(4\pi)^{\frac{n}{2}}} \int_0^{+\infty} \frac{\mathrm{vol}_g(B(x,\rho\sqrt{s}))}{\omega_n(\rho\sqrt{s})^n} e^{-\frac{\rho^2}{4}}\rho^{n+1}\, \mathrm{d}\rho \\ &= \frac{\omega_n}{2(4\pi)^{\frac{n}{2}}} \int_0^{+\infty} \mathcal{V}(x,\rho\sqrt{s}) e^{-\frac{\rho^2}{4}}\rho^{n+1}\, \mathrm{d}\rho. \end{aligned}$$

Therefore, since the volume ratio in monotone non-increasing with respect to the radius, the map $\lambda \mapsto \theta_x(\lambda s, 0)$ is monotone non-increasing as well. In [CMT24] we proved that the map $\lambda \mapsto \theta_x(\lambda t, \lambda s)$ can be seen as an interpolation between the monotonicity of the on-diagonal heat kernel ratio and the one of the volume ratio. More precisely, we obtained the following.

Proposition 3.17. *Let (M^n, g) be a complete manifold with $\mathrm{Ric}_g \geq 0$ and $x \in M$. Then the map $\lambda \mapsto \theta_x(\lambda s, \lambda t)$ is monotone non-increasing for $s \geq t$, monotone non-decreasing for $s \leq t$.*

The idea of the proof is to use Li-Yau inequality (LY0) and the properties of U coming from the ones of the heat kernel, in order to show that the derivative of $\lambda \mapsto \theta_x(\lambda s, \lambda t)$ has the same sign as $t - s$.

Monotone quantities depending on the Kato constant. In Corollary 5.10 of [CMT24] we used the Li-Yau inequality in order to show that if on a closed manifold (M^n, g) we have for some $T > 0$

$$k_T(M^n, g) \leq \frac{1}{16n}, \quad \int_0^T \frac{\sqrt{k_\tau(M^n, g)}}{\tau} \, \mathrm{d}\tau \leq \Lambda < +\infty$$

then there exists a function F tending to 1 as λ tends to 0 such that the map

$$\lambda \mapsto \theta_x(\lambda s, \lambda t) F(\lambda),$$

is monotone non-increasing if $t \leq s$ and non-decreasing otherwise. Thanks to [CMT23a] and the Li-Yau inequality (LY), we can improve this result to obtain:

Theorem 3.18. *Let (M^n, g) be a complete manifold such that*

$$k_T(M^n, g) \leq \frac{1}{16n}, \quad \phi(t) = \int_0^t \frac{k_\tau(M^n, g)}{\tau} \, \mathrm{d}\tau \leq \Lambda < +\infty.$$

Then there exists a constant c_n and for any $s, t > 0$ there exists $\overline{\lambda} = \overline{\lambda}(s, t, \Lambda, n)$ such that the map

$$\lambda \in [0, \overline{\lambda}] \mapsto \theta_x(\lambda s, \lambda t) \exp\left(c_n \phi(\tau) \left(\frac{t}{s} - \frac{s}{t}\right)\right) \tag{3.17}$$

is monotone non-increasing if $t \leq s$, non-decreasing if $t \geq s$.

With respect to the original statement in [CMT24], we include here complete manifolds and we do not have to consider the square root of the Kato constant in the definition of ϕ. The key point in the proof of Theorem 3.18 is to show the appropriate differential inequality, as in the case of non-negative Ricci curvature. We set $\Gamma_\tau = e^{5k_\tau(M_n, g)} - 1$ and prove that for any $t > 0, \tau \in (0, t)$ and $s \in (0, t/2\Gamma_\tau)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta_x + n\Gamma_\tau \left(\frac{t}{s} - \frac{s}{t}\right) \theta_x,$$

has the same sign as $t - s$ (see [CMT24, Proposition 5.8]). The fact that Γ_τ appears in the second summand is the reason why the function ϕ appears in the monotonicity formula (3.17), see the proof of [CMT24, Corollary 5.10] for the details.

A direct consequence of the previous theorem and of the relation (3.16) between θ_x and $\mathcal{H}$ is the following monotonicity for the on-diagonal heat kernel ratio.

Corollary 3.19. *Let (M^n, g) be a complete manifold such that*

$$k_T(M^n, g) \leq \frac{1}{16n}, \quad \phi(t) = \int_0^t \frac{k_\tau(M^n, g)}{\tau} \,\mathrm{d}\tau \leq \Lambda < +\infty.$$

Then there exist constants $\eta \in (0,1), c_n > 0$ such that the map

$$t \in (0, \eta T] \mapsto \exp(c_n \phi(t)) \mathcal{H}(t, x) \tag{3.18}$$

is monotone non-decreasing.

Monotone quantities on non-collapsed strong Kato limits. We first observe that Θ and θ are well-defined on PI-Dirichlet spaces, and continuous respectively in measured Gromov-Hausdorff convergence and Mosco-Gromov-Hausdorff convergence (see Sections 5.1 and 5.2 in [CMT24]). Moreover, heat-kernel bounds on PI-Dirichlet spaces imply bounds on Θ and θ. In particular, the bounds on θ_x depend on the measure of balls around x, see [CMT24, Remark 5.6]. Kato limits being PI-Dirichlet spaces, all these properties apply. We would also like to obtain analogue monotone quantities as the ones in Theorem 3.18 and Corollary 3.19: yet, observe that the Kato bound is not enough to obtain this kind of result, because of the assumption on the function ϕ in Theorem 3.18. This was our main motivation to introduce strong Kato bounds. Indeed, when assuming a strong Kato bound (SK) on a sequence of complete manifolds $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha \in A}$ satisfying (Dyn), we can actually replace each function ϕ_α in (3.17) and (3.18) by

$$\Phi(t) = \int_0^t \frac{f(s)}{s} \,\mathrm{d}s,$$

to obtain the analogue statements on each manifold M_α of the sequence. Moreover, the non-collapsing assumption (NC) and the bounds on θ_x guarantees that for any point in the limit θ_x is finite and does not vanish. As a consequence, we obtain:

Theorem 3.20. *Let (X, d, μ, o) be a non-collapsed strong Kato limit. Then there exist a positive constant η and a positive increasing function $\upphi : (0, \eta T] \to \mathbb{R}$ tending to zero in zero and such that for any $x \in X$ the map*

$$t \in (0, \eta T] \mapsto \exp(\upphi(t)) \mathcal{H}(t, x) \tag{3.19}$$

is non-decreasing.

This allow to define an on-diagonal heat kernel density for each point $x \in X$ as follows:

$$\uptheta(x) = \lim_{t \to 0} \mathcal{H}(t, x) \in [1, +\infty). \tag{3.20}$$

The on-diagonal heat kernel density can be shown to be equal to the inverse of the volume density, defined by

$$\uprho(x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{\omega_n r^n} \in (0, 1]. \tag{3.21}$$

Observe that, while ρ is a local quantity, θ depends on the heat kernel, thus it is a global quantity. In [CMT24] we used the monotonicity of θ_x to prove that tangent cones of non-collapsed strong Kato limits are metric cones, then the properties of the volume density ρ in order to show volume continuity and stratification as stated in Theorem 3.8. In [CMT22], we used the properties of the on-diagonal heat kernel ratio and of θ to get Reifenberg regularity, the last point in Theorem 3.8. In the following, we briefly sketch the proofs of "tangent cones are metric cones" and of Riefenberg regularity.

Tangent cones are metric cones

For the details of the following proof we refer to the proof of Theorem 5.11 in [CMT24]. We recall two properties of Θ_x and θ_x (see Lemma 5.3 and Proposition 5.5 in [CMT24]):

1. On a metric measure space (X, d, x), Θ_x is constant equal to $c > 0$ if and only if for any $r > 0$ we have $\mu(B(x,r)) = c\omega_n r^n$.
2. On a PI-Dirichlet space $(X, \mathsf{d}_{\mathcal{E}}, \mu, \mathcal{E})$ we have for any $x \in X$ and $s > 0$

$$\Theta_x(s) = \lim_{t\to 0} \theta_x(s,t).$$

In order to show that a tangent cone $(Y, \mathsf{d}_Y, \mu_Y, x)$ of a non-collapsed strong Kato limit (X, d, μ, o) is a metric cone for any $x \in X$, it is enough to prove that Θ^Y_x is constant. Indeed, this implies that for any $r > 0$ we have $\mu_Y(B(x,r)) = \Theta^Y_x(1)\omega_n r^n$ and, since $(Y, \mathsf{d}_Y, \mu_Y, x)$ is a weakly non-collapsed $\mathrm{RCD}(0,n)$ space, we can apply a result of De Philippis and Gigli [DPG18, Theorem 1.1] to conclude that it is a metric measure cone. To show that Θ^Y_x is constant, we start with some observations. For any $x \in X$, there exists a sequence $x_\alpha \in M_\alpha$ converging to x and for any $t \in (0,T]$, $s > 0$ the continuity of θ_x with respect to Gromov-Hausdorff convergence implies that

$$\theta^X_x(s,t) = \lim_\alpha \theta^{M_\alpha}_{x_\alpha}(s,t).$$

For s and t fixed, Theorem 3.18 ensures that there exist constants ε and k such that the maps

$$\lambda \in (0,\varepsilon] \mapsto \theta^{M_\alpha}_{x_\alpha}(\lambda s, \lambda t)\exp(k\phi(\lambda t)),$$

are monotone, so the same is true for

$$\lambda \in (0,\varepsilon] \mapsto \theta^X_x(\lambda s, \lambda t)\exp(k\phi(\lambda t)).$$

Then we can define the limit of this map as λ tends to 0, denote it

$$\vartheta_x(s,t) = \lim_{\lambda\to 0} \theta^X_x(\lambda s, \lambda t).$$

Because of non-collapsing and heat-kernel bounds, this limit is finite and non-vanishing. Moreover, by construction the map $(s,t) \mapsto \vartheta_x(s,t)$ is 0-homogeneous. We show that on a tangent cone $(Y, \mathsf{d}_Y, \mu_Y, x)$ at x we have for any $s, t > 0$

$$\theta^Y_x(s,t) = \vartheta_x(s,t).$$

This is simply due to the fact that Y is the limit of rescalings of X, to the rescaling law for the heat kernel and the continuity of θ_x^Y. As a consequence, the map $(s,t) \mapsto \theta_x^Y(s,t)$ is 0-homogenous. Since

$$\Theta_x^Y(s) = \lim_{t\to 0} \theta_x^Y(s,t),$$

the map $s \mapsto \Theta_x^Y(s)$ is also 0-homogenous. Therefore, it is constant, and we can conclude that $(Y, \mathsf{d}_Y, \mu_Y, x)$ is a metric measure cone.

Reifenberg regularity

A consequence of [CMT24, CMT22] is that the regular set of a non-collapsed strong Kato limit (X, d, μ, x)

$$\mathcal{R} = \{x \in X \text{ s.t. there is a unique tangent cone } (\mathbb{R}^n, \mathsf{d}_e, \mathcal{H}^n, 0)\},$$

coincides with the set in which the on-diagonal heat kernel density θ, or the volume density ρ, are constant equal to 1 (see for example [CMT22, Proposition 5.5]). The rough idea to prove Reifenberg regularity is that if $\theta(x)$, or $\rho(x)$, are close enough to 1, then it is possible to construct bi-Hölder charts into $\mathbb{R}^n$ around x. There are two key steps in order to prove this fact.

1. First one shows the so-called Reifenberg property of balls: if $\theta(x)$ or $\rho(x)$ are close enough to 1, then all balls $B(y,r)$ contained in a unit ball around x are Gromov-Hausdorff close to a Euclidean ball of radius r.

2. Then the intrinsic Reifenberg theorem of J. Cheeger and T. H. Colding [CC97, Theorem A.1.1] implies the existence of bi-Hölder charts.

In the case of Ricci limits and using the volume density, J. Cheeger, W. Jiang and A. Naber gave an alternative proof of the latter point, based on a Transformation Theorem, see Theorems 7.2 and 7.10 and [CJN21].

In the case of non-collapsed strong Kato limits, we can use either the classical approach based on the volume density ρ, or show the Reifenberg property of balls based on θ. In [CMT22] we followed this path: we used the almost monotonicity of the on-diagonal heat kernel, Li-Yau inequality, and a rigidity result for the heat kernel that was proven in [CT22]. Because of the novelty of this second approach in the literature of Gromov-Hausdorff limits of manifolds, we chose to present it here.

Reifenberg regularity of balls. We are going to sketch the proof of the following result, that is detailed in [CMT22, Section 5].

Theorem 3.21. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if (X, d, o) is a non-collapsed strong Kato limit and for some $x \in X$, $t \in (0, \delta T)$ we have*

$$\mathcal{H}(t,x) \leq 1 + \delta,$$

then for any $y \in B(x, \sqrt{t})$ and $s \in (0, \sqrt{t}]$ we have

$$\mathsf{d}_{\mathrm{GH}}(B(y,s), \mathbb{B}^n(0,s)) \leq \varepsilon s.$$

Theorem 3.21 is a direct consequence of the corresponding statement for manifolds. For the sake of convenience, we introduce the following notation.

Definition 3.8. Let $n \in \mathbb{N}$, $n \geq 3$, $T > 0$ and $f : (0, T] \to \mathbb{R}$ a non-negative, non-decreasing function such that

$$f(T) \leq \frac{1}{16n}, \quad \lim_{t \to 0} f(t) = 0, \quad \int_0^T \frac{f(s)}{s} \, \mathrm{d}s \leq \Lambda < +\infty.$$

We denote by $\mathcal{SK}(n, f)$ the set of isometry classes of n-dimensional complete Riemannian manifolds (M^n, g) such that for all $t \in (0, T]$

$$k_t(M^n, g) \leq f(t) \tag{3.22}$$

For $v > 0$, we denote by $\mathcal{SK}(n, f, v)$ the set of isometry classes of manifolds in $\mathcal{SK}(n, f)$ such that for some $o \in M$ $\mathrm{vol}_g(B(o, \sqrt{T})) \geq v$.

With this notation, a non-collapsed strong Kato limit is an element of the closure $\overline{\mathcal{SK}(n, f, v)}$ with respect to Gromov-Hausdorff topology.

Theorem 3.22. *For any $\varepsilon > 0$ there exists $\nu > 0$ depending on f, n, ε such that if (M^n, g) belongs to $\mathcal{SK}(n, f)$ and for some $x \in M$, $t \in (0, T]$ we have*

$$k_t(M, g) \leq \nu, \quad \mathcal{H}(x, t) \leq 1 + \nu,$$

then for any $y \in B(x, \sqrt{t})$ and $s > 0$ such that $B(y, s) \subset B(x, \sqrt{t})$ we have

$$\mathsf{d}_{\mathrm{GH}}(B(y, s), \mathbb{B}^n(0, s)) \leq \varepsilon s. \tag{3.23}$$

The key point is to prove (3.23) for the ball $B(x, \sqrt{t})$, that is:

$$\mathsf{d}_{\mathrm{GH}}(B(x, \sqrt{t}), \mathbb{B}^n(0, \sqrt{t})) \leq \varepsilon \sqrt{t}. \tag{3.24}$$

We briefly explain why. By heat kernel convergence, if $\mathcal{H}(x, t)$ is close to one, then for $y \in B(x, \sqrt{t})$ the quantity $\mathcal{H}(y, t)$ is close to 1 as well [CMT22, Corollary 5.11]. Moreover, an easy consequence of Corollary 3.19 is that for any $\delta \in (0, 1)$ one can choose ν small enough such that if $k_t(M, g) \leq \nu$, then for any $y \in M$ and $s \in (0, t]$ we have $\mathcal{H}(y, s) \leq \mathcal{H}(y, t)(1 + \delta)$, see [CMT22, Lemma 5.7]. Theorem 3.22 then follows by combining this information and applying (3.24) with a different center and radius. We are then left to prove (3.24) under the assumptions of Theorem 3.22.

Sketch of the proof of (3.24). We refer to [CMT22, Theorem 5.9] for the details of the proof, which goes by contradiction. There exists $\varepsilon_0 > 0$ such that for any sequence $\{\delta_\ell\}$ tending to zero we can find a sequence of manifolds (M_ℓ, g_ℓ) and points $x_\ell \in M_\ell$, such that up to re-scaling we have

$$k_1(M_\ell, g_\ell) \leq \delta_\ell, \quad \mathcal{H}(x_\ell, 1) < 1 + \delta_\ell,$$

and

$$\mathsf{d}_{\mathrm{GH}}(B(x_\ell,1),\mathbb{B}^n(0,1)) \geq \varepsilon_0. \tag{3.25}$$

It is possible to show that an upper bound on $\mathcal{H}(1,x_\ell)$ implies a lower bound $v(n)$ only depending on n on the volume of the ball $B(x_\ell,1)$. As a consequence, for any ℓ the pointed manifold (M_ℓ,g_ℓ,x_ℓ) belongs to $\mathcal{SK}(n,f,v(n))$ and we can extract a subsequence converging to a non-collapsed strong Kato limit (X,d,x). We also have $\mathcal{H}(1,x)\leq 1$, but since $\mathcal{H}(1,x)\geq 1$ we get $\mathcal{H}(1,x)=1$. Because of (3.25), we have

$$\mathsf{d}_{\mathrm{GH}}(B(x,1),\mathbb{B}^n(0,1)) > \varepsilon_0.$$

We aim to contradict the latter inequality and show that (X,d) is isometric to the Euclidean space $(\mathbb{R}^n,\mathsf{d}_e)$. For that, we prove that the heat kernel H of $(X,\mathsf{d},\mathcal{H}^n,\mathsf{Ch})$ is Euclidean, that is for any $z,y\in X$ and $t>0$

$$H(t,z,y)=(4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{\mathsf{d}(z,y)^2}{4t}\right). \tag{3.26}$$

Theorem 1.1 of [CT22] then ensures that (X,d) is isometric to $(\mathbb{R}^n,\mathsf{d}_e)$. Recall Varadhan's formula:

$$\mathsf{d}(x,y)^2=\lim_{\sigma\to 0}U(\sigma,x,y),$$

where $U(\sigma,x,y)=-4\sigma\log((4\pi\sigma)^{\frac{n}{2}}H(\sigma,x,y))$. We aim to show that for any $s\in(0,1]$ and $y\in X$ we have

$$U(s/4,x,y)=U(s/2,x,y), \tag{3.27}$$

so that

$$U(s/2,x,y)=\lim_{\sigma\to 0}U(\sigma,x,y)=\mathsf{d}(x,y)^2.$$

This implies (3.26) for $x\in X$, for any $s\in(0,1]$ and $y\in X$. It is not difficult to extend this result to all $s>0$ and $y,z\in X$ by using the monotonicity of $\mathcal{H}$ and the appropriate estimate on the derivatives of the heat kernel (see Step 3 in the proof of [CMT22, Theorem 5.9]). In order to prove (3.27), we introduce:

$$\phi_x(s,y)=(4\pi s)^{\frac{n}{2}}H^2(s/2,x,y).$$

A simple computation shows that (3.27) is equivalent to

$$\phi_x(s,y)=H(s/4,x,y). \tag{3.28}$$

Let L be the self-adjoint operator associated to the Cheeger energy of $(X,\mathsf{d},\mathcal{H}^n)$. By the definition of the heat kernel, the function $(s,y)\mapsto H(s/4,x,y)$ is the unique solution to the equation

$$\left(4\frac{\partial}{\partial t}+L\right)u=0,$$

satisfying

$$\lim_{s\to 0}u(s,\cdot)=\delta_x(\cdot).$$

We observe that heat kernel bounds and non-collapsing imply respectively that for any $s > 0$ and $y \in X \setminus \{x\}$ we have:

$$\lim_{\mathsf{d}(x,y)\to\infty} \phi_x(s,y) = 0, \quad \lim_{s\to 0} \phi_x(s,y) = 0.$$

Moreover, a simple computation ensures that for any $s > 0$.

$$\int_X \phi_x(s,y)\,\mathrm{d}\mathcal{H}^n(y) = \mathcal{H}(s,x).$$

We know that $\mathcal{H}(1,x) = 1$. Besides, by using [CMT22, Lemma 5.7] and the fact that the Kato constants of the converging manifolds tend to 0, we obtain that the map $s \to \mathcal{H}(x,s)$ is monotone non-decreasing. Then for any $s \in (0,1]$ we have $\mathcal{H}(s,x) = 1$ and, as a consequence, for any $s \in (0,1]$

$$\int_X \phi_x(s,y)\,\mathrm{d}\mathcal{H}^n(y) = 1.$$

Therefore,

$$\lim_{s\to 0} \phi_x(s,\cdot) = \delta_x(\cdot).$$

Moreover, the classical Li-Yau inequality (LY0) holds on $(X, \mathsf{d}, \mathcal{H}^n)$ (see Proposition 2.9 and Remark 2.10 in [CMT22]). Therefore, with some computations, we also obtain

$$\left(4\frac{\partial}{\partial t} + L\right)\phi \geq 0.$$

But we have

$$\int_X \phi_x(s,y)\,\mathrm{d}\mathcal{H}^n(y) = \int_X H(s/4,x,y)\,\mathrm{d}\mathcal{H}^n(y) = 1,$$

thus we get (3.28), which leads to (3.27) and to the conclusion of the proof. □

Existence of bi-Hölder charts. In [CMT22] we gave a quantitative version of Cheeger-Colding's Intrinsic Reifenberg theorem that can be stated as follows and that, together with a covering argument, directly leads to the last point in Theorem 3.8.

Theorem 3.23. *Let (X, d, o) be a non-collapsed strong Kato limit. For any $\alpha \in (0,1)$ there exists $\delta > 0$ such that for any $x \in X$ satisfying $\theta(x) < 1+\delta$ there exist $r \in (0, \sqrt{T})$ and a homeomorphism $u : B(x,r) \to u(B(x,r)) \subset \mathbb{R}^n$ such that for any $y, z \in B(x,r)$ we have*

$$\alpha r^{1-\frac{1}{\alpha}} \mathsf{d}(y,z)^{\frac{1}{\alpha}} \leq |u(y) - u(z)| \leq \frac{1}{\alpha}\mathsf{d}(y,z)^{\alpha} r^{1-\alpha}.$$

Instead of giving a sketch of the proof, in this section we aim to illustrate the tools that we used and the main differences with the existing literature for Ricci limits. For the details, we refer the interested reader to Sections 3 and 5.4 of [CMT22].

A first tool is given by *harmonic almost splittings.* These maps were introduced in the study of Ricci limits and extensively used both when dealing with limits of smooth manifolds and in the context of RCD spaces, see for instance [CN15, CJN21, Bam20, BPS21]. We refer to [CMT22, Definition 3.3] for the precise definition of a harmonic splitting map on a Kato limit, and focus on the definition on manifolds.

Definition 3.9. Let (M^n, g) be a complete manifold, $x \in M$, $\varepsilon, r > 0$ and $k \in \mathbb{N}$. A *harmonic* (k, ε)*-almost splitting* is a harmonic map $u : B(x,r) \to \mathbb{R}^k$ such that $\|du\|_{L^\infty(B(x,r))} \leq 2$ and for the Gram matrix G_u defined by $(G_u)_{i,j} = \langle du_i, du_j \rangle$ we have

$$\fint_{B(x,r)} \|G_u - \mathrm{Id}_k\| \,\mathrm{d}\,\mathrm{vol}_g < \varepsilon, \tag{3.29}$$

where Id_k is the identity $k \times k$-matrix and the norm is defined by

$$\|A\|^2 = \sup\{{}^t(A\xi)A\xi,\ \xi \in \mathbb{R}^k \text{ such that } {}^t\xi\xi = 1\}.$$

Whenever $k = n$, harmonic almost splittings can be considered as "almost harmonic charts", because, in average, their Gram matrix is close to the identity. Observe that the definition of an almost splitting often includes a condition on the Hessian:

$$r^2 \fint_{B(x,r)} |\nabla du|^2 \,\mathrm{d}\,\mathrm{vol}_g < \varepsilon^2.$$

However, under a lower Ricci bound or a Kato bound, this inequality can be deduced from (3.29) by using Bochner formula: see [CMT24, Proposition 3.5].

The importance of harmonic almost splitting maps is that, roughly speaking, their existence is equivalent to Gromov-Hausdorff closeness to a Euclidean ball: GH-closeness to a ball in $\mathbb{R}^k$ implies the existence of a harmonic almost splitting in $\mathbb{R}^k$; under the appropriate assumptions, the existence of a harmonic almost splitting in $\mathbb{R}^k$ implies GH closeness to a Euclidean ball, see Section 3.2 and Corollary 5.13 in [CMT22]. In order to prove this under a Kato bound, we used the appropriate results for convergence in energy of harmonic functions on converging sequences of PI-Dirichlet spaces: for the details we refer to the appendices of [CMT24, CMT22]. We point out that all of these convergence results do not depend on any curvature condition, while previous convergence results for harmonic functions were obtained in the more restrictive setting of RCD spaces, for instance [AH17, AH18].

The key point in proving Theorem 3.23 is to show that whenever $\theta(x)$ is close to 1, harmonic almost splitting are bi-Hölder homeomorphisms. It is actually enough to prove this on manifolds, that is, to show:

Theorem 3.24. *There exists* $\varepsilon_0 \in (0,1)$ *depending on* f, n *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $\eta \in (0,1)$ *there exists* $\delta > 0$ *depending on* f, n, ε, η *such that if* $(M^n, g) \in \mathcal{SK}(n, f)$*,* $x \in M$ *and* $t \in (0, \sqrt{T}]$ *satisfy*

$$k_t(M^n, g) < \delta, \quad \mathcal{H}(x,t) \leq 1 + \delta,$$

then any harmonic (n, δ)*-almost splitting* $u : B(x, \sqrt{t}) \to \mathbb{R}^n$ *with* $u(x) = 0$ *is a diffeomorphism between* $B(x, (1-\eta)\sqrt{t})$ *onto its image. Moreover, for any* $y, z \in B(x, (1-\eta)\sqrt{t})$ *we have*

$$(1-\varepsilon)\frac{\mathsf{d}_g(y,z)^{1+\varepsilon}}{(\sqrt{t})^\varepsilon} \leq |u(y) - u(z)| \leq (1+\varepsilon)\mathsf{d}_g(y,z),$$

As in the case of the proof of the Canonical Reifenberg Theorem of J. Cheeger, W. Jiang and A. Naber [CJN21, Theorem 7.10], the previous statement is a consequence of the Reifenberg regularity of balls given by Theorem 3.22, the properties of almost splitting maps and a transformation theorem, see [CMT22, Theorem 5.14]. The rough idea of the transformation theorem is the following. In general, a harmonic (k, ε)-almost splitting u on a ball $B(x, 1)$ is not a harmonic (k, ε)-almost splitting on balls of smaller scale $B(x, s)$ for $s \in (0, 1)$, because of the average on balls in (3.29): one gets a worse estimate with the volume of $B(x, s)$ in the denominator. But under the appropriate assumptions, one can find a transformation matrix $T_{x,s}$ such that the map $T_{x,s} \circ u$ is a harmonic (k, ε)-almost splitting on $B(x, s)$. This property, together with a control on the norm of $T_{x,s}$, allows to obtain the bi-Hölder regularity of u. In addition to the ones of [CN15] and [CJN21], several versions of the transformation theorem have been proven in the literature, for instance in the work of Q. S. Zhang and M. Zhu [ZZ19], in the setting of manifolds with bounded Bakry-Émery tensor, or in the work of R. Bamler [Bam20], which applies to the study of singularities of the Ricci flow with bounded scalar curvature. In all of these articles, the proofs of the transformation theorem are done by contradiction. For instance, in [CJN21], the transformation theorem for manifolds with a Ricci lower bound relies on a precise spectral gap for limit cones, which improves the previous work of C. Ketterer concerning RCD cones [Ket15] and implies a specific growth for harmonic functions on cones. Contradicting the statement of the transformation theorem leads to the existence of a harmonic linear splitting on balls of smaller size, therefore to the conclusion.

The main difference with respect to the previous proofs, is that in our case, under the weaker assumption of a strong Kato bound, we give a direct proof that only relies on the cited above results for the convergence of harmonic functions (see [CMT22, Theorem 3.8]) and on elementary properties of harmonic maps in the Euclidean space. In particular, our proof does not depend on RCD theory. We refer to Section 5.4 in [CMT22] for the details.

3.7 A geometric application

This section is devoted to presenting the main results of [CMT23b], which extend the torus stability proven by J. Cheeger and T. H. Colding to manifolds with a small Kato constant. Results of G. Carron [Car19] and C. Rose [Ros19] show that an analogue of Bochner theorem holds in this setting:

Theorem 3.25. *Let $n \in \mathbb{N}$, there exists $\delta(n) > 0$ such that if (M^n, g) is a closed manifold of diameter D such that $k_{D^2}(M^n, g) \leq \delta(n)$, then its first Betti number satisfies $b_1(M) \leq n$.*

In Proposition 4.1 and Remark 4.2 of [CMT23b] we also gave an alternative proof of this result. G. Carron had raised the following question in [Car19] what happens if the Kato constant is small enough and the first Betti number is equal to n? Our Theorem 3.9 answers this question by stating that the manifold is GH-close to a flat torus, and moreover diffeomorphic to a flat torus if a strong Kato bound holds. In the following, we present the main ideas of our proof.

We took inspiration from the proof proposed by S. Gallot in his Bourbaki seminar [Gal98] in the case of almost non-negative Ricci curvature: instead of using harmonic approximations of Busemann functions as in Colding's original argument, he showed that the so-called Albanese map is a GH-almost isometry between the manifold and a flat torus. The Intrinsic Reifenberg theorem of Cheeger and Colding then allows to get the diffeomorphism. G. Gallot also conjectured that the Albanese map itself is a diffeomorphism.

Let (M^n, g) be a closed manifold of diameter D and first Betti number equal to n. We refer to [CMT23b, Section 4] for the precise definitions of the Albanese map $\mathcal{A} : M \to \mathbb{R}^n/\mathbf{\Gamma}$ and of its lift $\widehat{\mathcal{A}} : \widehat{M} \to \mathbb{R}^n$, where $\widehat{M}$ is the Abelian covering of M. As in S. Gallot's proof, we showed that, if the Kato constant is small enough, then the Albanese map $\mathcal{A}$ is a GH-almost isometry. To do so we rely on the following steps.

1. By using the appropriate estimates for harmonic forms [CMT23b, Proposition 4.1], we prove that $\widehat{\mathcal{A}}$ is a harmonic almost splitting on a ball of large radius whenever the Kato constant of the manifold is small enough.

2. We show that if the Kato constant is small enough, a harmonic almost splitting into $\mathbb{R}^n$ is actually a GH-almost isometry. Observe that this result is stated in [CMT23b, Theorem 3.1] for closed manifolds, then extended to a normal covering of residually finite deck transformation group. Thanks to [CMT23a], we can directly prove [CMT23b, Theorem 3.1] for complete manifolds, and apply it to $\widehat{\mathcal{A}}$ restricted to the appropriate ball.

3. We then follow S. Gallot's argument to show that $\mathcal{A}$ itself is a GH-almost isometry (see [CMT23b, Section 5]).

In order to prove the existence of the diffeomorphism in case of a strong Kato bound, we prove that $\mathcal{A}$ is the required diffeomorphism, answering S. Gallot's question under a weaker assumption with respect to almost non-negative Ricci curvature. This is an application of our previous result on Reifenberg regularity. More precisely, we need two main steps.

1. We show that there exists $\hat{o} \in \widehat{\mathcal{A}}$ such that the on-diagonal heat kernel ratio $\mathcal{H}(4D^2, \hat{o})$ is close to one;

2. We adapt the Reifenberg regularity result [CMT22, Theorem 5.19] to normal coverings with residually finite deck transformation group to obtain that $\widehat{\mathcal{A}}$ is a diffeomorphism from a large enough ball onto its image (see [CMT23b, Proposition 3.4]). Again, thanks to [CMT23a], we can now directly apply the new version of Reifenberg regularity on complete manifolds stated in Theorem 3.22.

The fact that the Albanese map $\mathcal{A}$ is a diffeomorphism then follows because it is a local diffeomorphism and a finite cover of the torus, and the torus is finitely cover by tori only.

The usual Reifenberg results under almost non-negative Ricci curvature rely on a control on the volume ratio. In our case, we use the on-diagonal heat kernel ratio instead. In the following, we focus on briefly presenting the proof the first step, that is, of the estimate on the on-diagonal heat kernel. We refer to the proof of [CMT23b, Claim 6.1] for the details. The precise statement is given by:

Proposition 3.26. *Let $f : [0,1] \to \mathbb{R}$ be a non-negative, non-decreasing function such that*

$$\int_0^1 \frac{f(s)}{s}\,\mathrm{d}s < \infty.$$

Then there exist $\eta(n,f) > 0$ and $\delta(n,f) \in (0, \eta(n,f)]$ such that if (M^n, g) is a closed manifold of diameter D satisfying

$$b_1(M) = n,\ k_{D^2}(M^n, g) \leq \delta(n,f) \text{ and } k_{tD^2}(M^n, g) \leq f(t) \text{ for all } t \in (0,1],$$

and if $\widehat{M}$ is its Abelian covering with heat kernel $\widehat{H}$ then there exists $\hat{o} \in \widehat{M}$ such that

$$\widehat{\mathcal{H}}(4D^2, \hat{o}) \leq 1 + \eta(n,f),$$

where $\widehat{\mathcal{H}}(t,x) = (4\pi t)^{\frac{n}{2}} \widehat{H}(t,x,x)$.

Sketch of the proof. We introduce on $\widehat{M}$ an almost Euclidean heat kernel, that is, for any $\varepsilon \in (0,1)$, $x, y \in \widehat{M}$ and $t > 0$

$$\mathbb{H}_\varepsilon(t,x,y) := \frac{1}{(1+\varepsilon)(4\pi t)^{n/2}} \exp\left(-(1+\varepsilon)\frac{\mathsf{d}^2_{\hat{g}}(x,y)}{4t}\right). \tag{3.30}$$

The desired result follows from the almost monotonicity of the on-diagonal heat kernel combined with two estimates: an upper bound on $\mathbb{H}_\varepsilon(t,x,y)$ for any $t > 0$, $x, y \in \widehat{M}$ and a lower estimate on the integral over $\widehat{M}$ of $\mathbb{H}_\varepsilon(t, \hat{o}, \cdot)$ for $t \geq D^2$. We briefly explain how we obtained these two bounds.

Step 1. For any integer $\ell \geq 4$ we show that there exists $\delta_1(n, f, \varepsilon, \ell)$ such that if

$$k_{D^2}(M^n, g) \leq \delta \leq \delta_1(n, f, \varepsilon, \ell),$$

then for any $x, y \in \widehat{M}$ and $t \in (0, \ell D^2]$

$$\mathbb{H}_\varepsilon(t,x,y) \leq \widehat{H}(t,x,y). \tag{3.31}$$

For this, we use the Li-Yau inequality (LY) and a similar argument to the one illustrated in the proof of Proposition 3.15, where we obtained inequality (3.12). This leads to

$$\frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-F(\ell D^2)\frac{\mathsf{d}^2_{\hat{g}}(x,y)}{4t}\right) \exp\left(-C(n)\int_0^{\ell D^2} \frac{k_s(M^n,g)}{s}\,\mathrm{d}s\right) \leq \hat{H}(t,x,y), \tag{3.32}$$

where F is an explicit function depending on the Kato constant, that can be made smaller than $(1+\varepsilon)$ and $C(n)$ is a dimensional constant. The strong Kato bound ensures that the second exponential term in the previous expression can be made smaller than $(1+\varepsilon)^{-1}$, which leads to (3.31).

Step 2. We prove that for any $\varepsilon \in (0,1)$ and $t > D^2$,

$$\int_{\widehat{M}} \mathbb{H}_\varepsilon(t,\widehat{o},y)\,\mathrm{d}\nu_{\widehat{g}}(y) \geq \frac{1-C(n)\sqrt[3]{\delta}}{(1+\varepsilon)^{\frac{n}{2}+1}}\left(1-C(n)\left(\frac{D}{\sqrt{t}}\right)^{n+2} - C(n)e^{-\frac{D^2}{5\sqrt[3]{\delta}t}}\right).$$

The proof of this lower estimate relies on direct computations and a Euclidean lower bound on the volume of a ball $B(\hat{o},r)$ for $r \in [D,\delta^{-\frac{1}{6}}D]$ due to the fact that $\widehat{\mathcal{A}}$ is a harmonic almost splitting. This volume bound was originally show in [CC00a, Theorem 1.2]: we noticed that the lower Ricci bound that is assumed in the original statement is not needed in the proof, and reformulated the argument in the proof of [CMT23b, Theorem 7.1]. □

3.8 Perspectives

We present below some future research directions related to our study of Kato limits.

Codimension 4

M. Anderson conjectured in the 1990s that the limit of a non-collapsed sequence of n-manifolds for which the Ricci tensor is bounded has only codimension 4 singularities of locally finite $\mathcal{H}^{n-4}$-Hausdorff measure. The codimension 4 conjecture was proven by J. Cheeger and A. Naber [CN15], and the locally finiteness of the singularities by W. Jiang and A. Naber in [JN21]. It is natural to ask which kind of Kato or integral bound may lead to a similar result. In an on-going work with G. Carron and D. Tewodrose, we will show that a Morrey control of the Ricci tensor, and not only its negative part, implies that the limit space has only codimension 4 singularities. In particular, we develop the tools that allow us to quantitatively show, with direct proofs, some fundamental results: an epsilon-regularity theorem, a transformation theorem, the appropriate estimates for harmonic functions.

Geometric applications of codimension 4

We plan to apply the above codimension 4 result to prove geometric consequences on manifolds of dimension 4. It is well-known that the geometry and topology of a compact, smooth 4-manifold (M,g) are related by the Chern-Gauss-Bonnet formula, which involves on one side the Euler characteristic of the manifold, and on the other side the integrals of the square norm of the Weyl tensor and of the Q-curvature. The Q-curvature is a geometric quantity depending on the scalar and Ricci curvatures, that can be seen as the conformal analogue of the Gauss curvature of surfaces. Thanks to [CGY03, CGZ20], some rigidity results are known under the control of a conformal invariant: the ratio $\beta(M,[g])$ between the integral of $|Weyl|^2$ and the one of the Q-curvature. In particular, when M admits a metric of positive Yamabe constant and Q-curvature, [CGZ20] conjectures that if the infimum of $\beta(M, [g])$ over all conformal classes is equal to 4, then the manifold is diffeomorphic to $\mathbb{CP}^2$. The proof of these rigidity results often consists in choosing an appropriate conformal representative, using it as a starting point for the Ricci flow, then extracting a converging sequence from this flow. For a sequence of minimizing conformal classes, in collaboration with G. Carron we intend to choose instead conformal representatives with positive scalar curvature and positive Q-curvature. These metrics satisfy the appropriate Morrey control, which will allow us to apply our results to obtain the convergence to a space with isolated conical singularities. We plan to prove that the limit metric is Bach flat and that the singular set is empty. We will then have to show the appropriate rigidity statement for Bach flat metrics.

Examples and pathological Kato limits

A natural question consists in finding explicit examples of sequences of manifolds satisfying a Kato bound, but not an L^p bound. Together with D. Tewodrose, we intend to build a metric on a manifold such that the Kato constant is bounded by a Kato potential that does not belong to L^p. Besides, one can make sense of measures as Kato potentials. We plan to build a singular Kato measure on a manifold and show that it can be approached in Gromov-Hausdorff topology by smooth metrics satisfying a Kato bound, giving an example of Kato limit.

In another line of study, we recovered a large part of Cheeger-Colding theory, but do Kato limits have exactly the same regularity as Ricci limits? In a recent work, we constructed an example of a 2-dimensional strong, non-collapsed Kato limit which, in contrast with Ricci limits and RCD spaces, contains branching geodesics and is essentially non-branching. This shows that there exist pathological Kato limits whose regularity is worse than the one of Ricci limits. Together with D. Semola, we intend to push this study further: we plan to show that there are examples for which tangent cones do not vary Hölder continuously along a minimizing geodesic. This property has been shown in [CN12] for Ricci limits, thanks to Abresch-Gromoll inequality and Laplacian comparison. These tools are not available in the case of Kato bounds: this leads to believe that the Hölder continuity of tangent cones can be a specific characteristic of Ricci limits.

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